Foundation Course
in
Higher Mathematics

Foundation Course in Higher Mathematics

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Dedication

Chief (Dr) Edwin Kiagbodo Clark, (OFR) to whom this book is dedicated was born in the 1st quarter of the 20th Century, at Kiagbodo Village in the then Old Delta Province of Western Region of Nigeria. Over the years, Chief Clark has made his mark as an elder statesman, educationist, administrator, a detribalized bridge-builder, and a leader of no mean genre. He had also served as the voice of the voiceless of the impoverished people of Niger Delta, which had over the decades of Nigeria's independence, served and continues to serve as the petroleum and gas bearing region that produces over 90% of the national income.

At several fora in both national and international, Chief Clark has been an advocate of Minority Rights, and true fiscal federalism in Nigeria since 1962 when he was made the Executive Secretary to the then Zikist Movement in London. In particular, he has untirelessly championed the cause of youth's empowerment in the Niger Delta via education, skills acquisition, employment, and all forms of motivation to correct the misinformation and Machiavellian manipulation of the beneficiaries of the old order. Equally Chief Clark has promoted throughout the period of his stewardship, educational programmes across the length and breadth of Nigeria and beyond, which has earned him a prestigious national honour of the Federal Republic of Nigeria and that of the Republic of Togo.

The dedication of this 'seed' of technology is therefore the author's own little way to immortalize **Chief (Dr) Edwin Kiagbodo Clark, (OFR)** as a living Legend of our time. This is however a departure from our age-long tradition of recognizing the dead at the expense of the living.

You will never be tired Sir!

"The Niger Delta Constitutes the Soul of Nigeria and the liberation of Niger Delta is the liberation of Nigeria."
-Rev. Jesse Jackson

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This book has been greatly improved by the comments, critisms and advice of various academic icons, who have been kind enough to read part or all of the manuscript. I should like to thank most profoundly Professor Grace A. Alele-Williams – former Vice Chancellor University of Benin, Professor M.O. Adewale of University of Abuja, Professor S. Funso Akande of University of Jos, and Professor Oyewusi Ibidapo-Obe – Vice Chancellor University of Lagos for their invaluable assistance in this respect. Equally of mention, is Mr T.M. Asiru of the Department of Mathematics, Oyo State College of Education, Oyo State. Their efforts and contributions did much to improve the quality of the manuscript, but I, of course, remain responsible for its final form. Therefore, I encourage ideas in the form of comments or suggestions of any kind from users of this book that will help to improve its revised edition. You can reach me at **emmaohwadua@yahoo.com** or through the Publisher – University Press Plc., Ibadan; your contributions shall be highly acknowledged.

I wish to also express my appreciation to Pastor Darius A. Alhassan of National Mathematical Centre, Abuja for the initial typesetting of the manuscript. I am equally grateful to William Peter, Ayuba Daniel Gumwos, Mr Awanfi O. Adams, Mr Peter Udosen, Mrs Omonye Ikwenobe, Mr J.Babatunde Soares, Mr Texen Egbedi, Comrade Joe Angodeme Evah and a host of others, whose contributions, one way or the other made it possible for the publication of this text.

I am also extremely grateful to Dr (Mrs) Cecilia Ibru, MD/CEO of

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My appreciation also goes to my wife, Juliana and my kids – Fejiro, Obaro, and Kome – who waited patiently in the womb before arriving at the eve of publishing this text.

Above all, I give God almighty the glory for the successful completion of this epoch task.

E. Obaro Ohwadua
November 2006

Forward

Mathematics is a subject that is not popular among the majority of school pupils and students. At the tertiary level, it is hoped that those who study it are interested and that they require to understand basic mathematics in their profession. It is necessary to note that the subject initially developed because various civilizations recognized and used the concept of number and measurement in their daily affairs. Over time, man's inventiveness and better cognition and measurement and their applications have lead various civilizations and societies to apply these to relationships, planning, strategy formulation and greatest rigour in development of solutions for human and societal problems.

The applications of mathematics are very broad and spread into all areas of our civilization yet the subject is not exactly liked and sometimes unduly dreaded by us. In our country this may be the result of lack of well written books, the paucity of teachers this book at least begins to address one of our problems.

There is a general dearth of locally written textbooks for students in our tertiary institutions. This scarcity is even worse in subject areas like mathematics. Book-publishing companies in Nigeria are not keen on publishing books written by local authors for the tertiary level because they believe that it is not profitable for them to do so.

The scarcity of locally written and published books has resulted in students in our tertiary institutions to patronize books from India because they are usually cheaper than those from Europe or the

United States of America. Low priced editions of books are possible in India because Shri Pandi Jawaharal Nehru, the first Prime Minister, after India's independence in 1947 recognizing the role of books in a multicultural country like India set up a book trust and introduced partnership with book publishing houses in the United Kingdom. This partnership maintained copyright laws but at the same time helped to produce books for the mass of the people and for the schools. He promoted Hindi language for India but also ensured that people were sufficiently empowered to write books to promote the Indian culture. Thus, the state promoted a book industry that was absolutely necessary for the promotion of education. We could even now really take a leaf from that initiative for our country.

The author of this book, Mr. Emmanuel Obaro Ohwadua, has put together his lecture notes for over a decade, revised them and put them in a book form. The books covers very many topics. It includes a lot of what is required in mathematics in the first two years of University, Polytechnic, and Colleges of Education. It should meet the needs of student studying Engineering, Business, Medicine and of course Education.

In 13 chapters, the author treats tertiary level mathematics with dexterity that would give the student a better understanding of mathematics with relative ease. He treats topics like Number Systems; Equations; Real Sequences and Series; Theory of Mathematical Inductions; Combinatorics; Elementary Trigonometry; Theory of Complex Numbers; Matrix Algebra and Systems of Linear Equations; Vector Space and Subspace; Vectors; Logic and Logical Applications.

By writing this book, Mr. Ohwadua has thrown a challenge to his fellow lecturers in tertiary institutions in the country. I believe that if many more lecturers could follow his example and put their lecture notes in their various subjects, spanning many years together, it will be possible to persuade our local book publishers to publish the books for our tertiary institutions. Such books will help in solving the problems of handouts, which is very common in our tertiary institutions. It will also help in spurring many of our lecturers to write books and in generating a reading culture among our youths in tertiary institutions across the country. But the challenge must be accepted especially by Federal Universities to encourage their staff to cooperate and write texts for their students. This may reverse the tide of handout culture.

It is my hope that Mr. Ohwadua's publication is just a first edition. He will not only follow this first edition with improvements on subsequent editions within the next several months but will continue to develop further, some of the topics in this text for higher levels.

I highly recommend the book to those who have compulsory mathematics in their degree requirements and above all, to every pure science student in tertiary institutions in our country.

Grace A. Alele-Williams OFR, Ph.D
Professor of Mathematics Education (Lagos)
Former Vice Chancellor University of Benin
1 November, 2006

Preface

This course-book is written to add a spark to a subject that seems to scare students from mere hearing and sighting of a mathematics text, that 'the taste of the pudding is in the eating' – that mathematics remains one of the most fascinating subject to study. In support of this fact therefore, I have tried throughout in the course of writing this book to present topics in order of easy comprehension rather than in a strictly logical manner. Most topics are initiated with simple illustrations that demonstrate the basic ideas and the conceptual background of any new terminology; and each new term is illustrated by as many familiar and practical examples as possible. This seems especially important at an elementary treatise, because it serves to emphasize the fact that the abstract concepts all arise from the representation and description of the behaviour of real-world situations, with a view to solving some problem which has arisen or otherwise improving on that situation.

The text covers all the elementary courses in basic algebra, trigonometry, matrix, vectors and logic at the foundation levels in the university, polytechnic, and colleges of education as well as advanced level mathematics in high school. This is one of the most striking characteristics of the text because it combines the foundation level mathematics course contents of National Universities Commission (NUC), National Board for Technical Education (NBTE) and National Commission for Colleges of Education (NCCE). So, whichever tertiary institution you find yourself, insofar as you are in the sciences or engineering this text is your companion! It is equally meant for teachers/lecturers as it consists of well paced illustrations and varied examples that present various practical situations and applications. The chapters are arranged in such a way that gives the reader a progressive understanding, and the contents in each topic, are displayed in a sequential and simplistic manner that makes it easy for you to follow and comprehend.

This edition consists of thirteen chapters; beginning from Chapter 1 with Number Systems, which include Natural Numbers, Real Numbers, Integers, Rational and Irrational Numbers, and Elementary Number Theory. This is followed by Chapter 2, which deals with basic topics in the Theory of Equations including Linear (Inequalities, Simultaneous) and Non-Linear (Quadratic) Equations. Chapters 3 to 6 attempts to introduce various elementary mathematical theories such as Indices, Remainder and Factor Theorem, Logarithms and Surds – their basic laws and applications in simplifying algebraic expressions were adequately treated. Others, are Basic Set Theory, Relations, Real Sequences and Series, and Theory of Mathematical Induction. Next, is Chapter 7 which includes Factorial Notation, Permutations, Combinations and Binomial Theorem – various applications were considered particularly the application of Binomial Theorem in the expansion of expressions and approximations. Elementary Trigonometry and Theory of Complex Numbers were dealt with in Chapters 8 and 9, which consists of various definitions, manipulations and applications of Trigonometric Functions and Identities.

The subject of Matrices (Chapter 10) and their application is one of great importance; we have endevoured to give it full discourse without allowing it to overshadow the other topics. It is desirous however, to discuss not merely the formal operations with matrices but also the interpretation of the matrices by Linear Transformations. We have kept this geometric interpretation continually in view by emphasizing the appropriate properties of Vector Spaces in Chapter 11 that follows. The basic concepts and manipulations of Vectors and their applications to the solution of engineering problems were considered in Chapter 12. The last Chapter, Logic, is intended to enable students develop precise logical and abstract thinking ability to understand basic rules of Mathematical Logic and their applications to mathematical proofs and Circuit Theory. This chapter is particularly important at this stage because it introduces the student to embrace the study of mathematics not only in the abstract form but also its applications to real-life situations.

It is hoped that this text would be invaluable to any student who wants to be ahead of others.

E. Obaro Ohwadua
November 2006

Contents

Chapter 1

Number Systems

In the historical development of mankind, numerals were one of the first scientific feat recorded in ancient times to assist man in quantitative reasoning. At the beginning of civilization, the development of numerals was borne out of the quest to count objects, and carry out reckoning or calculations in all spheres of human endeavours - from astrology to commerce, and social activities.

Roman numerals were first developed and used until the tenth century B.C. as symbols in arithmetical notation. The basic letters are I(1), V(5), X(10), L(50), C(100), D(500), and M(1000), and intermediate as well as higher numbers are formed according to the laid down rules - these rules can be found in any elementary text on arithmetic notation. Following the Roman numerals, are Arabic numerals whose symbols are 0,1,2,3,4,5,6,7,8,9 and based on the decimal system which have been in general use since the tenth century. Afterwards, various numerical representation and computations have evolved, which are laid down in computational rules that are now reduced to a few logical and fundamental concepts and axioms that form the Number Systems.

Suffice to say that people learn to play around with numbers from elementary school; nevertheless, the inclusion of this chapter in a book of this nature is to provide adequate theoretical framework for a better understanding of the fundamental principles and concepts of what you have already known at college level. Therefore, what we shall be discussing in this chapter is not entirely new, but to further expose and prepare you for a very interesting subject of mathematics.

1.1 Natural Numbers

A number is an idea. Each whole number is simply an idea that we associate with certain groups, or sets – for now we shall refer to 'set' or 'group' as a number of things belonging together and having the same attributes. For example, a set of male students in a class, refers to the number of male students in that class.

The concept of natural numbers originated from the necessity to count objects. All whole numbers beginning with 1 arranged in increasing order, form a series (ordered arrangement) of natural numbers and we shall denote this series or set by the letter N, so that

$$\mathbb{N} = \{1, 2, 3, 4, ..., 10, ...\}$$

1.2 Integers

An integer is a whole number; a number that is not a fraction. Integers include the set of natural numbers including zero, and their respective additive inverses. We shall denote this set by the letter $\mathbb{Z}$, so that;

$$\mathbb{Z} = \{..., -3, -2, 0, 1, 2, 3, ...\}$$

where the set of negative integers is denoted by $\mathbb{Z}^-$ while the set of positive integers denoted by $\mathbb{Z}^+$. We shall also call the familiar set of natural numbers, positive integers. Algebraic laws of real numbers discussed in section 1.4.1 apply equally to the set of integers.

1.3 Rational Numbers

A rational number is a number which can be expressed as an integer or as a quotient of integers. They are of the form a/b, where a is an integer and b is a natural number and is denoted by the letter

$\mathbb{Q}$. For example, 1/2, 5/3, −7. Another name for rational number is *fraction*.

Inductively, the word 'rational' is based on 'ratio'. Thus in a rational number like a/b, the number a is called the *numerator* and the number b is called the *denominator*. Observe that all integers are also examples of rational numbers since their denominator is 1.

1.4 Non-Rational or Irrational Numbers

Non-rational or irrational number is a number which cannot be expressed as an integer or the quotient of integers, as in π, $\sqrt{2}$, e^x (x being real). Rational and irrational numbers can be distinguished by their decimal representation. A rational number has a decimal expansion which eventually repeats itself:

$$\frac{1}{5} = 0.2\underline{0}, \quad \frac{2}{3} = 0.\underline{6}, \quad \frac{13}{15} = 0.8\underline{6}$$

The portion of the decimal which is underscored is understood to be repeated indefinitely in each case. (The reader should have no difficulty in showing that such repetitions must always occur when a rational number is written in its decimal expansion). This implies that every repeating decimal represents a rational number. Ordinarily it is not too easy to show that some particular number is irrational. There is no simple proof, for example, of the irrationality of e^π. Nevertheless, the irrationality of certain numbers such as $\sqrt{2}$ and $\sqrt{3}$ is not too difficult to establish. e.g.; $\sqrt{2} = 1.414\ 213\ 562...$, and $\pi = 3.141\ 592\ 654....$ Observe that they have no repeating decimal, hence $\sqrt{2}$ and $\sqrt{3}$ are irrational.

Lemma 1.4.1. *If n is an integer which is not a perfect square, then $\sqrt{n}$ is irrational or non-rational.*

Proof. Let $n = 16 \in \mathbb{Z}^+$, then $\sqrt{16} = 4$, therefore, $\sqrt{16}$ is a perfect square and is not irrational, but rational.
Conversely, let $n = 5 \in \mathbb{Z}^+$; then, $\sqrt{5} = 2.236\ 067\ 978...$; and $\sqrt{5}$ is not a perfect square; therefore, $\sqrt{5}$ is irrational. □

1.5 Real Numbers

A real number is any rational or irrational number that does not contain a even root of a negative number. All numbers that falls between $-\infty$ to $+\infty$ including zero (0) are referred to as real numbers – natural numbers, integers, rational numbers and irrational numbers.

The following diagram or scale displays all the components/members of real numbers;

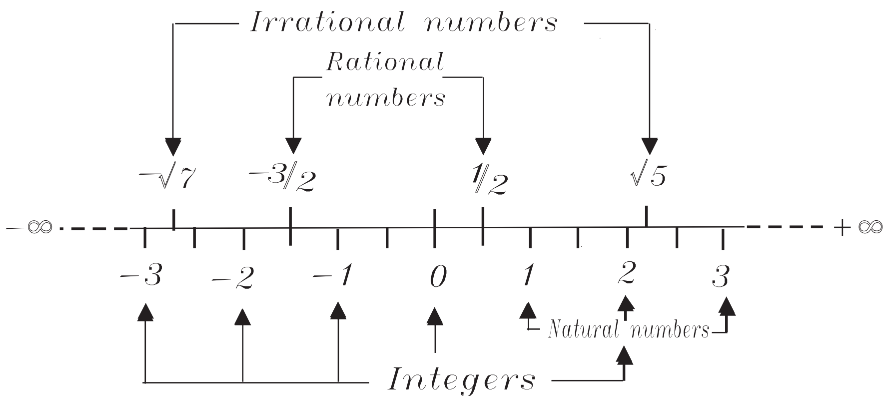

Figure 1.1: Representation of Real Numbers on a Scale

From the diagram above, the point $-\infty$ and $+\infty$ denotes an infinite number or quantity. It is the point or series of points on the line/scale that by supposition lie at an infinite distance from any definite point on the scale. Similarly, the numeral or symbol '0' at the centre of the scale is a cardinal number indicating the absence of quantity or the point where a continuous quantity changes its sign from plus to minus, or vice versa.

1.5.1 Laws of Algebra of Real Numbers

The entire set of numbers represented by decimals is called the set of *real numbers* and is denoted by the letter $\mathbb{R}$ such that given any real number say a, b, c, defined over the operation of multiplication $(*)$ and addition $(+)$, the following conditions which may be regarded as the laws of algebra of real numbers are satisfied:

(i) $\left.\begin{array}{rcl} a+b &=& b+a \\ a*b &=& b*a \end{array}\right\}$ commutative laws

For example,

$$2+16=16+2=18 \quad \text{and} \quad 2*16=16*2=32$$

(ii) $\left.\begin{array}{rcl} a+(b+c) &=& (a+b)+c \\ a*(b*c) &=& (a*b)*c \end{array}\right\}$ associative laws

For example,

$$2+(6+9)=(2+6)+9=17 \quad \text{and} \quad 2*(6*9)=(2*6)*9=108$$

(iii) $\left.\begin{array}{rcl} a*(b+c) &=& a*b+a*c \\ (a+b)*c &=& a*c+b*c \end{array}\right\}$ distributive laws

For example,

$$3*(6+9)=3*6+3*9=45 \quad \text{and} \quad (3+6)*9=3*9+6*9=81$$

(iv) $\left.\begin{array}{rcl} a+0 &=& a \\ a*1 &=& a \end{array}\right\}$ identity laws

For example,

$$8+0=8 \quad \text{and} \quad 8*1=8$$

(v) If $c \neq 0$ and $ca=cb$, then $a=b$ (cancelation law)

For example,

$$5*a=5*b \quad \Rightarrow \quad a=\frac{5*b}{5}=b \quad \text{i.e.} \quad a=b$$

(vi) $a+(-a)=0$ (additive inverse)

For example,

$$10+(-10)=0$$

(vii) If a and b are two real numbers with $a \neq 0$, then there exists a real number c such that $ac=b$. This c is denoted by b/a; the number a/a is denoted by 1. We write a^{-1} for $1/a$ if $a \neq 0$, such that

$$a*a^{-1}=1 \qquad \text{(multiplicative inverse)}$$

For example,

$$4*4^{-1}=1$$

1.5.2 Principles of Order

If a and b are different real numbers, then either a is greater than b ($a > b$) or a is less than b ($a < b$). In other words, the relation $<$ or $>$ establishes an ordering among the real numbers. For example, let $a = 11$, and $b = 30$;   then, $11 < 30$ and $30 > 11$

1.5.3 Geometrical Representation of Real Numbers

The real numbers can be represented geometrically as points on a line (*the real axis*) denoted by $\mathbb{R}$. A point is selected to represent 0 and another point to represent 1 as in Figure 1.1; and these points determine the scale.

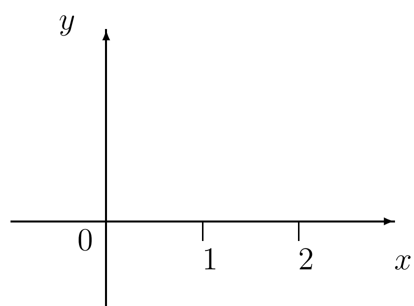

Figure 1.2: Points on a real axis

Each point on the real axis in Figure 1.1, $x \in \mathbb{R}$ corresponds to one and only one real number and, conversely, each real number is represented by a single point. Similarly, ordered pair of real numbers can be represented by points in the coordinate plane. Therefore, the set of ordered pairs of real numbers is called the *number plane*; any ordered pair being referred to as a point of the number plane. We shall similarly denote the number plane by the symbol $\mathbb{R}^2$. Just as in the case of the *number line*, we may apply the geometrical terminology to the number plane; for instance, the set of pairs $(x, y) \in \mathbb{R}^2$ or points whose coordinates satisfy the equation $y = x$ is a straight line as in Figure 1.2.

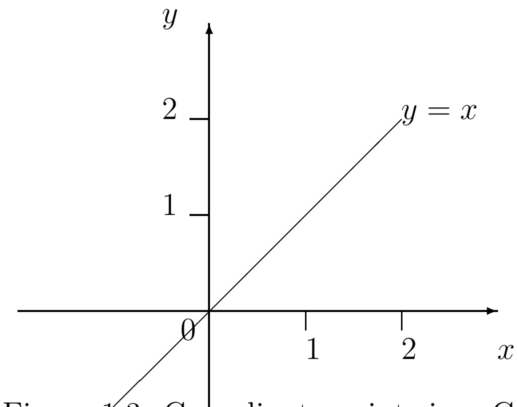

Figure 1.3: Co-ordinate points in a Cartesian plane

1.5.4 Decimal Representation of Real Numbers

Every real number a, has a decimal expansion of the form;

$$a = \pm P.a_1a_2...a_n...;$$

where P is either 0 or a natural number and each a_i is one of the digits from 0 to 9. More generally, a number which has a decimal expansion which ends in zeros can be represented as for example; $1/8 = 0.125\ 000...$, while the number 2/3 can also be written as a decimal expansion which ends in six. Thus, we have 2/3=0.666 666... . Except for situations like this, decimal expansions are unique.

1.5.5 Absolute Value (or Modulus) of a Real Number

If a is any real number, then we define the absolute value (modulus) of a; denoted by $|a|$, as follows:

$$|a| = \begin{cases} a, & \text{if } a \geq 0 \\ a, & \text{if } a < 0 \end{cases}$$

For example if $a = 3$, then $|a| = 3$, if $a = -2.51$, then $|a| = 2.51$. In particular, absolute or modulus of a real number can be defined as;

$$|a| = \sqrt{a^2}$$

For example, let $a = 3$, then;

$$|3| = \sqrt{3^2} = \sqrt{9} = 3$$

Similarly, if $a = -3$, then;

$$|-3| = \sqrt{(-3)^2} = \sqrt{9} = 3$$

In other words, the absolute or modulus of any real number be it negative or positive gives a positive real number. Thus $|a| = 0$; iff $a = 0$ and $|a| \in \mathbb{R}^+$ if $a \neq 0$. The following rules hold, for $a, b \in \mathbb{R}$,

(i) $a, b \in \mathbb{R}$,
For example, let $a = 8$ and $b = -3$, then $8, -3 \in \mathbb{R}$

(ii) $|a \cdot b| = |a| \cdot |b|$,
For example,

$$|8 \cdot -3| = |8| \cdot |-3| = 24$$

(iii) $|a| - |b| \leq |a + b| \leq |a| + |b|$,
For example,

$$|-5| - |12| < |-5 + 12| < |-5| + |12| \quad \text{i.e.,} \quad -7 < 7 < 17$$

or

$$|12| - |-5| = |12 + (-5)| < |12| + |-5| \quad \text{i.e.,} \quad 7 < 17$$

(iv) $|a| - |b| \leq |a - b| \leq |a| + |b|$
For example,

$$|-5| - |12| < |-5 - 12| = |-5| + |12| \quad \text{i.e.,} \quad -7 < 17$$

1.6 Complex Number

A complex number is the square root of a negative real number, such as $\sqrt{-2}$, $\sqrt{-4}$, $\sqrt{-7}$ etc. Full details on complex number can be found in Chapter 9 of this book.

Chapter 2

Equations

Real life problems in the sciences, engineering or business are transformed into mathematical expressions by the use of equations, which can then be solved to obtain desired solutions. A mathematical expression is a group of characters, numbers, and symbols put together to represent a statement. Examples of mathematical expressions are $4x$, $\sin y$, $log_a z$, e^x etc.; the characters x, y, z in these examples are referred to as *variables*, while other values such as 4, a, e are referred to as constants. A variable is a quantity representing one of a group of objects that is susceptible of fluctuating in value or magnitude under different conditions.

When two expressions, say, E_1 and E_2 are linked together by the symbol of equality, '=', an equation ; $E_1 = E_2$ arises. For example, $5x + 3 = y$. Here, $5x + 3$ represents the left-hand-side (LHS), and y represents the right-hand-side (RHS) of the equation. An equation is therefore a statement of the equality or inequality of two mathematical expressions. It is broadly classified into two:

(i) Linear Equations

(ii) Non-Linear Equations

2.1 Linear Equations

Linear equations are mathematical expressions, containing one or more variables, of the first degree and can be represented graphically by straight line as in Figure 2.1.

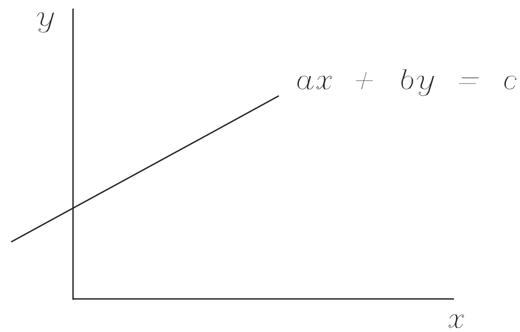

Figure 2.1: Linear graph

In other words, an equation which contains only one variable, say x and a constant is called a *Linear equation in one variable*, for example

$$ax + b = c \tag{2.1.1}$$

where a, b, and c are constants.

Example 2.1.1

Solve the equation,

$$\frac{x+1}{7} - \frac{3(x-2)}{14} = 1$$

Solution

Multiply both sides by 14 (the L.C.M.),

$$2(x+1) - 3(x-2) = 14$$

Remove the brackets,

$$2x + 2 - 3x + 6 = 14$$

Collect like terms;

$$\begin{aligned} 2x - 3x &= 14 - 6 - 2 \\ -x &= 6 \\ \therefore \quad x &= -6 \end{aligned}$$

Check. When $x = -6$, the L.H.S. of the equation becomes;

$$\frac{-6+1}{7} - \frac{3(-6-2)}{14} = -\frac{5}{7} + \frac{24}{14} = \frac{14}{14} = 1$$

Example 2.1.2

Solve the equation,

$$\frac{2y+1}{y-1} = \frac{6y+1}{3y-2}$$

Solution

Multiply both sides by $(y-1)(3y-2)$ (the L.C.M.);

$$(2y+1)(3y-2) = (6y+1)(y-1)$$

Remove the brackets;

$$6y^2 - 4y + 3y - 2 = 6y^2 - 6y + y - 1$$

Take away $6y^2$ from both sides;

$$-4y + 3y - 2 = -6y + y - 1$$

collect like terms;

$$\begin{aligned} -4y + 3y + 6y - y &= 2 - 1 \\ 4y &= 1 \end{aligned}$$

Divide both sides by 4 (the coefficient of y);

$$\therefore \qquad y = \frac{1}{4}$$

Check. When $y = \frac{1}{4}$

$$\begin{aligned} \text{L.H.S.} &= \frac{2(\frac{1}{4})+1}{1/4-1} \\ &= \frac{\frac{1}{2}+1}{\frac{1}{4}-1} \\ &= -2 \end{aligned}$$

$$\begin{aligned} \text{R.H.S.} &= \frac{6(\frac{1}{4})+1}{3(\frac{1}{4})-2} \\ &= \frac{\frac{3}{2}+1}{\frac{3}{4}-2} \\ &= -2 \end{aligned}$$

2.1.1 Linear Inequalities in One Variable

In this case, instead of having equality sign between two expressions, we now replace it with an inequality sign of the form $(<, >, \leq, \geq)$. The inequality notations are:

'$<$' means less than
'$>$' means greater than
'$\leq$' means less than or equal to
'$\geq$' means greater than or equal to

Just as was demonstrated in Examples 2.1.1 and 2.1.2 respectively, the same method can be applied to solve inequalities, but with a particular restriction when terms of both sides are multiplied or divided by a negative constant. However, if both sides are multiplied and divided by a negative constant, the sign of the inequality would change. More specifically, if;

$$a + c < b + c, \quad \text{for all } c, \tag{2.1.2}$$

$$ac < bc, \quad \text{for all positive } c \tag{2.1.3}$$

$$\frac{a}{c} < \frac{b}{c}, \quad \text{for all positive } c \tag{2.1.4}$$

$$\text{but } ac > bc, \quad \text{for all negative } c \tag{2.1.5}$$

$$\frac{a}{c} > \frac{b}{c}, \quad \text{for all negative } c \tag{2.1.6}$$

The two inequalities (2.1.5) and (2.1.6) have their signs changed; from the 'less than' notation to 'greater than' notation because the operation was carried out with negative constant, (i.e. $-c$).

Example 2.1.3

Find the range of values of x for which

$$\frac{x}{3}+\frac{x}{5}+\frac{x}{4}>1$$

Solution

Multiply both sides by 60 (the L.C.M.);

$$\begin{aligned} 20x+12x+15x &> 60 \\ 47x &> 60 \end{aligned}$$

Divide both sides by 47;

$$\therefore \qquad x>\frac{60}{47}$$

Example 2.1.4

Find the range of values of x for which, $4(2-x)<3(3+2x)$

Solution

Remove the brackets;

$$8-4x<9+6x$$

Collect the like terms;

$$\begin{aligned} 8-9 &< 6x+4x \\ -6x-4x &< 9-8 \quad \text{or} \quad -1<10x \\ -10x &< 1 \quad \text{or} \quad \frac{-1}{10}<x \end{aligned}$$

Divide both sides by 10;

$$\begin{aligned} -x &< \frac{1}{10} \\ \therefore \qquad x &> \frac{-1}{10} \end{aligned}$$

2.1.2 Simultaneous Linear Equations

A linear equation in two or more variables is an equation involving these variables in the first degree only. Thus, a linear equation in two variables such as $ax+$ by $=c$ has an infinite number of solutions. There is no unique solution.

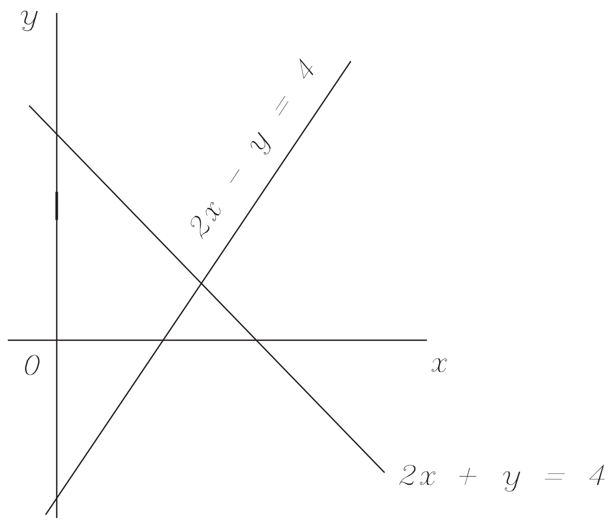

Figure 2.2: Graph of linear equations

Graphically (Figure 2.2), the equation represents a straight line (hence the linearity property) and the co-ordinates (x, y) of any point on the line satisfy the equation. However, if we are given two such equations at the same time, it becomes a *simultaneous equation* in two variable and a unique solution can be obtained if the two equations are independent in the two variables x and y. Note that two equations are said to be *linearly dependent* if one is a multiple of the other and graphically, the equations represent parallel lines superimposed on each other. In that case there is no unique solution. For example, the two equations $x-3y+2=0, 2x-6y+4=0$, are dependent as the second equation can be obtained by multiplying the first equation by 2.

There are three methods of solving simultaneous linear equations:

(a) Substitution Method

(b) Elimination Method

(c) Graphical Method

(a) Substitution Method

Example 2.1.5

Solve the equations, $2x + 3y = 10, 3x - 2y + 11 = 0$.

Solution

$$\begin{aligned} 2x + 3y &= 10 \qquad (i) \\ 3x - 2y &= -11 \qquad (ii) \end{aligned}$$

Isolate y in equation (i);

$$\begin{aligned} 3y &= 10 - 2x \\ y &= \frac{10 - 2x}{3} \qquad (iii) \end{aligned}$$

Substitute for y in (ii);

$$\begin{aligned} 3x - 2\left(\frac{10 - 2x}{3}\right) &= -11 \\ 3x - \left(\frac{20 - 4x}{3}\right) &= -11 \end{aligned}$$

Multiply both sides by 3 (the L.C.M.);

$$\begin{aligned} 9x - 20 + 4x &= -33 \\ 9x + 4x &= -33 + 20 \\ 13x &= -13 \\ x &= -\frac{13}{13} \\ &= -1 \\ \therefore \quad x &= -1 \end{aligned}$$

Substituting for x in (iii);

$$\begin{aligned} y &= \frac{10-2(-1)}{3} \\ &= \frac{10+2}{3} \\ &= \frac{12}{3} \\ \therefore \quad y &= 4 \end{aligned}$$

The solution is therefore, $x = -1$, $y = 4$.

Example 2.1.6

Solve the equation,

$$\frac{x}{2} + \frac{y}{3} = 5, \quad 2x - y = 6$$

.

Solution

$$\begin{aligned} \frac{x}{2} + \frac{y}{3} &= 5 \qquad (i) \\ 2x - y &= 6 \qquad (ii) \end{aligned}$$

Multiply equation (i) by 6 (the L.C.M.);

$$3x + 2y = 30 \qquad (iii)$$

Now, solving equations (ii) and (iii); first isolate y in equation (ii);

$$\begin{aligned} -y &= 6 - 2x \\ y &= 2x - 6 \end{aligned}$$

Substitute for y in (iii);

$$\begin{aligned} 3x + 2(2x-6) &= 30 \\ 3x + 4x - 12 &= 30 \end{aligned}$$

By rearrangement and collection of like terms;

$$\begin{aligned} 7x &= 30 + 12 \\ 7x &= 42 \\ \therefore \quad x &= \frac{42}{7} \\ &= 6 \\ \text{and} \quad y &= 2(6) - 6 \\ &= 12 - 6 \\ &= 6 \end{aligned}$$

$\therefore$ the solution is $x = 6$ and $y = 6$

(b) Elimination Method

Example 2.1.7

Solve the equation,

$$\frac{x+1}{y+1} = 2, \quad \frac{2x+1}{2y+1} = \frac{1}{3}$$

Solution

$$\begin{aligned} \frac{x+1}{y+1} &= 2 && (i) \\ \frac{2x+1}{2y+1} &= \frac{1}{3} && (ii) \end{aligned}$$

Multiply both equations by their respective L.C.M. and rearrange;

$$\begin{aligned} x + 1 &= 2(y + 1) \\ x + 1 &= 2y + 2 \\ x - 2y &= 2 - 1 \\ x - 2y &= 1 && (iii) \end{aligned}$$

Similarly from equation (ii);

$$\begin{aligned} 3(2x+1) &= 2y+1 \\ 6x+3 &= 2y+1 \\ 6x-2y &= 1-3 \\ 6x-2y &= -2 \end{aligned}$$

Dividing both sides by 2;

$$3x - y = -1 \qquad (iv)$$

Now, solving equations (iii) and (iv) we have;

$$\begin{aligned} x-2y &= 1 \\ 3x-y &= -1 \end{aligned}$$

Multiplying equation (iii) by 3 or equation (iv) by 2 - this is to produce equations in which the coefficients of x or y are equal. In choosing a multiplier at this stage, it is necessary to select the equation which will produce an easier and fast result.

However, taking the first multiplier of 3, then the above equations become;

$$\begin{aligned} 3x-y &= -1 \qquad (iv) \\ 3x-6y &= 3 \qquad (v) \end{aligned}$$

Now, taking away (v) from (iv) so that the terms containing x disappears;

$$\begin{aligned} 5y &= -4 \\ y &= -\frac{4}{5} \end{aligned}$$

Substituting for y in (iv);

$$\begin{aligned} 3x-(-\frac{4}{5}) &= -1 \\ 3x+\frac{4}{5} &= -1 \end{aligned}$$

Multiply through by 5;

$$\begin{aligned} 15x + 4 &= -5 \\ 15x &= -5 - 4 \\ x &= -\frac{9}{15} = -\frac{3}{5} \end{aligned}$$

$\therefore$ the solution is $x = -\frac{3}{5}, \quad y = -\frac{4}{5}$

Example 2.1.8

Solve the equation,

$$\frac{3}{x} + \frac{4}{y} = 2, \quad \frac{4}{x} - \frac{1}{y} = 3$$

.

Solution

$$\begin{aligned} \frac{3}{x} + \frac{4}{y} &= 2 \qquad (i) \\ \frac{4}{x} - \frac{1}{y} &= 3 \qquad (ii) \times 4 \end{aligned}$$

Multiplying equation (ii) by 4 and adding so that y disappears;

$$\begin{aligned} \frac{3}{x} + \frac{4}{y} &= 2 \qquad (i) \\ \frac{16}{x} - \frac{4}{y} &= 12 \qquad (iii) \\ \frac{3}{x} + \frac{16}{x} &= 14 \\ 3 + 16 &= 14x \\ \therefore \quad x &= \frac{19}{14} \end{aligned}$$

Now, substitute for x in (i);

$$\begin{aligned}\frac{3}{\frac{19}{14}} + \frac{4}{y} &= 2\\ \frac{3(14)}{19} + \frac{4}{y} &= 2\\ \frac{4}{y} &= 2 - \frac{42}{19}\end{aligned}$$

Multiply both sides by $19y$ (the L.C.M);

$$\begin{aligned}4(19) &= 2(19y) - 42y\\ 76 &= 38y - 42y\\ 76 &= -4y\\ \therefore \quad y &= -\frac{76}{4}\\ &= -19\end{aligned}$$

The solution is therefore $x = \frac{19}{14}, \quad y = -19$

Remark 1 *We shall however omit graphical method here, but interested readers can consult any elementary text.*

Further Examples

Example 2.1.9

Solve the equation:

$$\frac{a+1}{2} = \frac{2a+1}{3} = \frac{b+3}{4}$$

Solution

In this type of equation, it is advisable to reduce the equation into two separate equations of the form:

$$\frac{a+1}{2} = \frac{2a+1}{3} \quad \text{and} \quad \frac{2a+1}{3} = \frac{b+3}{4}$$

We can now solve for the values of a and b simultaneously using either method;

$$\begin{aligned} \frac{a+1}{2} &= \frac{2a+1}{3} && (i) \\ \frac{2a+1}{3} &= \frac{b+3}{4} && (ii) \end{aligned}$$

From equation (i);

$$\begin{aligned} 3(a+1) &= 2(2a+1) \\ 3a+3 &= 4a+2 \\ 3a-4a &= 2-3 \\ -a &= -1 \\ \therefore \quad a &= 1 \end{aligned}$$

From (ii);

$$\begin{aligned} 4(2a+1) &= 3(b+3) \\ 8a+4 &= 3b+9 \\ 8a-3b &= 9-4 \\ 8a-3b &= 5 && (iii) \end{aligned}$$

Substitute for a in (iii),

$$\begin{aligned} 8(1)-3b &= 5 \\ -3b &= 5-8 \\ -3b &= -3 \\ \therefore \quad b &= 1 \end{aligned}$$

Hence the solution is $a = 1, \quad b = 1$.

Example 2.1.10

Solve the equation,

$$\frac{x+2y+1}{4} = \frac{3x+y+1}{8} = \frac{2x+3y+2}{9}$$

Solution

If we resolve the equation further into two simpler equations we obtain;

$$\frac{x+2y+1}{4}=\frac{3x+y+1}{8} \quad \textit{and} \quad \frac{3x+y+1}{8}=\frac{2x+3y+2}{9}$$

Now, simplifying the first equation;

$$\begin{aligned} \frac{x+2y+1}{4} &= \frac{3x+y+1}{8} && (i) \\ 8(x+2y+1) &= 4(3x+y+1) \\ 8x+16y+8 &= 12x+4y+4 \\ 8x-12x+16y-4y &= 4-8 \\ -4x+12y &= -4 && (iii) \end{aligned}$$

Similarly, considering the second equation;

$$\begin{aligned} \frac{3x+y+1}{8} &= \frac{2x+3y+2}{9} && (ii) \\ 9(3x+y+1) &= 8(2x+3y+2) \\ 27x+9y+9 &= 16x+24y+16 \\ 27x-16x+9y-24y &= 16-9 \\ 11x-15y &= 7 && (iv) \end{aligned}$$

Solving equations (iii) and (iv), we obtain;

$$\begin{aligned} -4x+12y &= -4 && (iii)\times 5 \\ 11x-15y &= 7 && (iv)\times 4 \end{aligned}$$

Multiply equations (iii) by 5 and (iv) by 4 as above, so that the term y disappears;

$$\begin{aligned} -20x+60y &= -20 && (v) \\ 44x-60y &= 28 && (vi) \end{aligned}$$

$(v) + (vi)$ gives;

$$\begin{aligned} 24x &= 8 \\ \therefore \quad x &= \frac{8}{24} \\ &= \frac{1}{3} \end{aligned}$$

Substitute for x in (iii);

$$\begin{aligned} -4\left(\frac{1}{3}\right) + 12y &= -4 \\ -4 + 36y &= -12 \\ 36y &= -12 + 4 \\ 36y &= -8 \\ \therefore \quad y &= -\frac{8}{36} \\ &= -\frac{2}{9} \end{aligned}$$

The solution is therefore, $x = \frac{1}{3}, \quad y = -\frac{2}{9}$

Remark 2 *Linear equations in three variables x, y, z will be of the general form $ax + by + cz = d$ where $a, b, c, d,$ are constants. Although a series of three independent equations in three variables can be solved simultaneously but it is advisable to use Matrix Method for rapid solutions. Matrix Method for solving systems of linear equation is treated in Chapter (10).*

2.2 Non-Linear Equations

A non-linear equation is a mathematical expression involving any number of variables with degree greater than 1, and can be represented graphically by a smooth curve (Figure 2.3). Examples of non-linear equations include quadratic and polynomial equations.

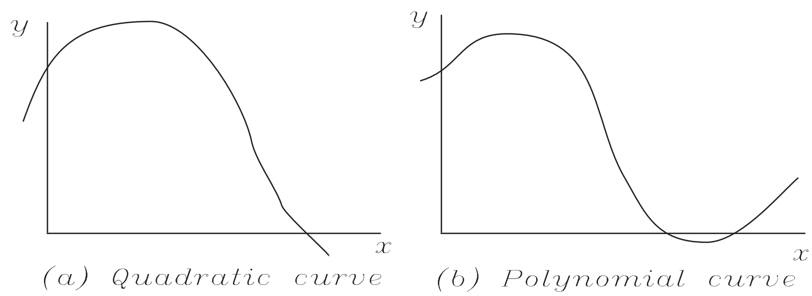

Figure 2.3: Graph of non-linear equations

2.2.1 Theory of Quadratic Equations

A quadratic equation in a single variable x is an equation of the second degree in x and is of the form $ax^2 + bx + c = 0$, where a, b, c are constants and $a \neq 0$. The L.H.S. expression i.e. $ax^2 + bx + c$ is referred to as a *quadratic function*. A second-degree quadratic equation in one variable always has two solutions or roots, a third-degree equation (cubic) in one variable always has three solutions, and so on.

There are three methods of solving quadratic equations. These are:

(a) Solution by factorization

(b) Method of completing the square

(c) Graphical method

We shall however omit the graphical method in this book and concentrate on the two analytical methods only. The graphical method can be read up in any elementary text in algebra.

(a) Solution by Factorization

In attempting the solution of any quadratic equation in a single variable x, it is important to try first the method of factorization. This entails the initial clearing of fractions (if any) by multiplying each term of the equation by the L.C.M. of the denominators of the fractions. Next, all the terms are transferred to the left hand side (L.H.S.) of the equation and like terms collected, so that the equation will now appear in the form.

$$ax^2 + bx + c = 0 \tag{2.2.1}$$

Then the L.H.S. is factorized where possible in the form $AB = 0$, and each of the linear factors i.e. A and B in x is equated to zero. Thus the method uses the principle that

$$\text{If } AB = 0 \text{ then either } A = 0 \text{ or } B = 0$$

giving a solution (or root) of the given equation. The roots obtained could be verified by substituting back the values in the original equation.

Example 2.2.1

Solve the equation $x^2 - 14x + 24 = 0$.

Solution

Observe that the factors of 24 could be (3,8); (2,12); (6,4); e.t.c., however, we search for the factors of 24 such that their addition gives you -14 (i.e. the coefficient of x). In this case the factors of 24 that could fit into that condition is (-2,-12), while the factors of x^2 is (x, x). We may decide to represent this diagrammatically in Figure2.4 as follows:

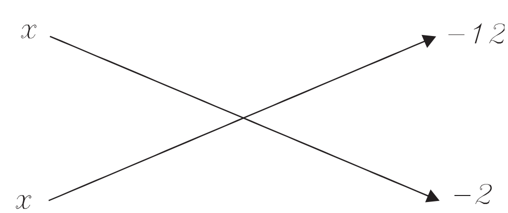

Figure 2.4:

The factors of the equation now become;

$$(x - 2)(x - 12) \quad = \quad 0$$

Which is simply put as;

$$\begin{aligned} x-2 &= 0 \quad \text{or} \quad x-12=0 \\ \therefore \quad x &= 2 \quad \text{or} \quad 12 \end{aligned}$$

Thus, the roots of the equation are (2,12).

Example 2.2.2

Solve:

$$\frac{x+7}{x+1}+\frac{2x+6}{2x+1}=5$$

Solution

Multiply through the given equation by the L.C.M. $(x+1)(2x+1)$ to clear the fractions;

$$(x+7)(2x+1)+(2x+6)(x+1) = 5(x+1)(2x+1)$$

$$\begin{aligned} \text{i.e., } (x+7)(2x+1)+(2x+6)(x+1)-5(x+1)(2x+1) &= 0 \\ (2x^2+x+14x+7)+(2x^2x+8x+6)-5(2x^2+x+2x+1) &= 0 \\ 2x^2+15x+7+2x^2+8x+6-10x^2-15x-5 &= 0 \\ \therefore \quad -6x^2+8x+8 &= 0 \end{aligned}$$

Dividing through by -2;

$$3x^2-4x-4=0 \qquad (i)$$

Factorizing (i);

$$\begin{aligned} (3x+2)(x-2) &= 0 \\ 3x+2 &= 0 \quad \text{or } x-2=0 \\ 3x &= -2 \quad \text{or} \quad x=2 \\ x &= -\frac{2}{3} \quad \text{or} \quad x=2 \end{aligned}$$

$\therefore$ the roots of the equation are $-\frac{2}{3}$, 2.

Check: If $x = -\frac{2}{3}$;
Then substituting for x in the equation;

$$\begin{aligned} \frac{-\frac{2}{3}+7}{-\frac{2}{3}+1} + \frac{2(-\frac{2}{3})+6}{2(-\frac{2}{3})+1} &= 5 \\ 19 - 14 - 5 &= 0 \end{aligned}$$

Similarly the reader can check for $x = 2$ and compare the result.

(b) Solution by Completing the Square Method

When a quadratic equation which has been manipulated into the form $ax^2 + bx + c = 0$, cannot be solved by factorization, then the method of completing the square can be used either directly or by the formula obtained as follows: Consider for example the general quadratic equation (2.2.1)

$$ax^2 + bx + c = 0, \quad a \neq 0$$

Rearrange to have; $ax^2 + bx = -c$
Divide through by the coefficient of x^2 (i.e. a);

$x^2+\frac{b}{a}x = -\frac{c}{a}$\{ which is the standard form of a quadratic equation\}

Completing the square of the L.H.S. by adding to each side the square of half the coefficient of x;

$$\begin{aligned} x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 &= -\frac{c}{a} + \left(\frac{b}{2a}\right)^2 \\ \left(x + \frac{b}{2a}\right)^2 &= -\frac{c}{a} + \frac{b^2}{4a^2} \\ &= \frac{b^2 - 4ac}{4a^2} \end{aligned}$$

Taking the square root of both sides;

$$\begin{aligned} x + \frac{b}{2a} &= \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} \\ &= \frac{\pm\sqrt{b^2 - 4ac}}{2a} \end{aligned}$$

Simplifying, we have;

$$\begin{aligned} 2ax + b &= \pm\sqrt{b^2 - 4ac} \\ 2ax &= -b \pm \sqrt{b^2 - 4ac} \\ \therefore \qquad x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \end{aligned}$$

This is often referred to as the '*almighty formula*'. Observe that a quadratic equation has two roots:

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

These two roots can be any of the following three types obtained in a quadratic equation. Consider the solution,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{D}}{2a}$$

$D = b^2 - 4ac$ is called the *discriminant* or *characteristic* of the quadratic equation. From the formula, three cases of the discriminant can be established which gives rise to the three types of roots obtained in a quadratic equation as follows:

(i) Real and distinct roots if $b^2 - 4ac > 0$

(ii) Real and equal roots if $b^2 - 4ac = 0$
In this case the solution becomes $x = \frac{-b \pm \sqrt{0}}{2a} = \frac{-b}{2a}$ (i.e. repeated roots)

(iii) Complex roots i.e. $b^2 - 4ac < 0$
In this case the discriminant D is negative, the solution is of the form;

$$x = \frac{-b \pm \sqrt{-D}}{2a} \quad (i.e. \quad b^2 - 4ac = -D)$$

The search for this solution gives rise to *complex numbers*. The subject of complex numbers shall not be discussed in this present section, however, in the event of encountering the square root of negative number, the reader may replace it by the product of i and the square root of the positive equivalent. In this case $i = \sqrt{-1}$. For now, we shall restrict ourselves to this level of knowledge. For example, take $D = -16$ then;

$$\begin{aligned}
\pm\sqrt{(-D)} &= \pm\sqrt{-16} \\
&= \pm\sqrt{(-1)16} \\
&= \pm 4i
\end{aligned}$$

Example 2.2.3

Solve the equation, $7x^2 - 2x - 3 = 0$ using the method of completing the square.

Solution

Rearranging, we have;

$$7x^2 - 2x = 3$$

Dividing through by the coefficient of x^2 (i.e. 7);

$$x^2 - \frac{2}{7}x = \frac{3}{7}$$

Completing the square;

$$\begin{aligned}
x^2 - \frac{2}{7}x + \left(\frac{2}{14}\right)^2 &= \frac{3}{7} + \left(\frac{2}{14}\right)^2 \\
x^2 - \frac{2}{7}x + \left(\frac{1}{7}\right)^2 &= \frac{3}{7} + \left(\frac{1}{7}\right)^2 \\
\left(x - \frac{1}{7}\right)^2 &= \frac{21+1}{49} \\
\left(x - \frac{1}{7}\right)^2 &= \frac{22}{49}
\end{aligned}$$

Taking the square root of both sides;

$$\begin{aligned}
x - \frac{1}{7} &= \pm\frac{\sqrt{22}}{7} \\
x &= \frac{1}{7} \pm \frac{\sqrt{22}}{7} \\
x_1 &= \frac{1}{7} + \frac{\sqrt{22}}{7}, \quad x_2 = \frac{1}{7} - \frac{\sqrt{22}}{7} \\
\therefore \qquad x_1 &= 0.81, \quad x_2 = -0.53
\end{aligned}$$

Example 2.2.4

Solve the equation using the formula,

$$x(x+2)(x+3) = (x-1)(x+4)(x+5)$$

Solution

Open the brackets and collect like terms;

$$\begin{aligned}
x(x^2 + 3x + 2x + 6) &= (x^2 + 4x - x - 4)(x+5) \\
x^3 + 5x^2 + 6x &= x^3 + 3x^2 - 4x + 5x^2 + 15x - 20 \\
x^3 + 5x^2 + 6x &= x^3 + 8x^2 + 11x - 20
\end{aligned}$$

$$\therefore \qquad x^3 - x^3 + 5x^2 - 8x^2 + 6x - 11x + 20 = 0$$

Thus,

$$\begin{aligned}
-3x^2 - 5x + 20 &= 0 \\
3x^2 + 5x - 20 &= 0 \qquad (i)
\end{aligned}$$

Invoking the formula, $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$; here, $a = 3, b = 5$ and $c = -20$. So that;

$$\begin{aligned}
x &= \frac{-5 \pm \sqrt{25 - 4(3)(-20)}}{2(3)} \\
&= \frac{-5 \pm \sqrt{265}}{6} \\
x_1 &= \frac{-5 + \sqrt{265}}{6}, \quad x_2 = \frac{-5 - \sqrt{265}}{6} \\
\therefore \qquad x_1 &= 1.88, \quad x_2 = -3.55
\end{aligned}$$

Example 2.2.5

Using the formula, solve the equation,

$$\frac{1}{x+1} + \frac{2}{x+3} = \frac{2}{3}$$

Solution

Multiply both sides by the L.C.M. $3(x+1)(x+3)$;

$$\begin{aligned} 3(x+3)+6(x+1) &= 2(x+1)(x+3) \\ 3x+9+6x+6 &= 2(x^2+4x+3) \\ 9x+15 &= 2x^2+8x+6 \\ 2x^2+8x-9x+6-15 &= 0 \\ 2x^2-x-9 &= 0 \qquad\qquad (i) \end{aligned}$$

Invoking the formula, $x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$, here, we have $a = 2$, $b = -1$ and $c = -9$;

$$\begin{aligned} x &= \frac{1 \pm \sqrt{1-4(2)(-9)}}{2(2)} \\ &= \frac{1 \pm \sqrt{73}}{4} \\ \therefore \quad x_1 &= \frac{1+\sqrt{73}}{4}, \quad x_2 = \frac{1-\sqrt{73}}{4} \\ \text{i.e.,} \quad x_1 &= 2.39, \quad x_2 = -1.89 \end{aligned}$$

Theorem 2.2.1. *If α and β $(\alpha > \beta)$ be the roots of the quadratic equation $ax^2+bx+c=0$, $a \neq 0$, to prove that the sum of roots $(\alpha+\beta) = -\frac{b}{a}$, and the product of roots, $\alpha\beta = \frac{c}{a}$.*

Proof. From the previous formula;

$$\begin{aligned}
\alpha &= \frac{-b+\sqrt{b^2-4ac}}{2a}, \quad \beta = \frac{-b-\sqrt{b^2-4ac}}{2a} \\
\therefore \quad \alpha+\beta &= \left(\frac{-b+\sqrt{b^2-4ac}}{2a}\right) + \left(\frac{-b-\sqrt{b^2-4ac}}{2a}\right) \\
&= \frac{-b+\sqrt{b^2-4ac}-b-\sqrt{b^2-4ac}}{2a} \\
&= -\frac{2b}{2a} \\
\therefore \quad \alpha+\beta &= -\frac{b}{a}
\end{aligned}$$

Similarly,

$$\begin{aligned}
\alpha\beta &= \left(\frac{-b+\sqrt{b^2-4ac}}{2a}\right)\left(\frac{-b-\sqrt{b^2-4ac}}{2a}\right) \\
&= \frac{(-b)^2-(\sqrt{b^2-4ac})^2}{4a^2} \\
&= \frac{b^2-(b^2-4ac)}{4a^2} \\
\therefore \quad \alpha\beta &= \frac{b^2-b^2+4ac}{4a^2} \\
&= \frac{4ac}{4a^2} \\
&= \frac{c}{a}
\end{aligned}$$

Therefore, the sum of roots, $\alpha+\beta = -\frac{b}{a}$,
and the product of roots, $\alpha\beta = \frac{c}{a}$ □

Remark 3 *If it be required to form an equation whose roots are α_1 and β_1 where α_1 and β_1 are functions of the above roots α and β, the required equation will be of the form.*

$$x^2 - (\textit{sum of roots})\, x + (\textit{product of roots}) = 0 \tag{2.2.2}$$

$$\Rightarrow \quad x^2 - (\alpha_1+\beta_1)x + \alpha_1\beta_1 = 0 \tag{2.2.3}$$

and the quickest method usually is to find both $(\alpha_1 + \beta_1)$ *and* $\alpha_1\beta_1$ *in terms of* $(\alpha+\beta)$ *and* $\alpha\beta$*, and then use the above results to obtain the sum and product of roots in terms of* a, b, c*. Thus, the values got will be substituted into the above equation (2.2.3) and the result simplified so that there are no fractions involved.*

Example 2.2.6

Find the sum and product of roots of the equation,

$$\frac{x+1}{2x-1} = \frac{2x-4}{x+2}$$

Solution

Multiply both sides by the L.C.M.; $(2x-1)(3x+2)$;

$$\begin{aligned} (x+1)(3x+2) &= (2x-4)(x-1) \\ 3x^2+5x+2 &= 2x^2+2x+4 \\ \therefore \quad x^2+3x-2 &= 0 \qquad (i) \end{aligned}$$

From the equation (i), $a = 1$, $b = 3$ and $c = -2$
Thus, the sum of roots,

$$\begin{aligned} (\alpha+\beta) &= -\frac{b}{a} \\ &= -\frac{3}{1} \\ &= -3 \end{aligned}$$

Similarly, the product of roots,

$$\begin{aligned} \alpha\beta &= \frac{c}{a} \\ &= -\frac{2}{1} \\ &= -2 \end{aligned}$$

Example 2.2.7

If α and β are the roots of the equation, $2x^2-5x+2=0$, find their sum and product.

Solution

From the equation, $a = 2, b = -5,$ and $c = 2$
The Sum of roots,

$$\begin{aligned}(\alpha+\beta) &= -\frac{(-5)}{2}\\ &= \frac{5}{2}\end{aligned}$$

The product of roots,

$$\alpha\beta = \frac{c}{a} = \frac{2}{2} = 1$$

Example 2.2.8

The sum of the roots of a quadratic equation is $\frac{17}{20}$ and the product is $\frac{-3}{20}$. Find the quadratic equation.

Solution

The required quadratic equation is;

$$x^2 - (\text{Sum of roots})x + (\text{Product of roots}) = 0$$

$$\begin{aligned}\text{i.e.,}\quad x^2 - \frac{17}{20}x + \left(\frac{-3}{20}\right) &= 0\\ x^2 - \frac{17}{20}x - \frac{3}{20} &= 0\end{aligned}$$

Multiplying through by the L.C.M (20);

$$20x^2 - 17x - 3 = 0$$

which is the required equation.

Example 2.2.9

If α and β are the roots of the equation $x^2 - 2x = 3$, find the values of $\alpha^2 + \beta^2$, $\alpha^3 + \beta^3$ and hence form the equations whose roots are (i) α^2 and β^2, (ii) α^3 and β^3

Solution

Rearrange the equation in standard form;

$$x^2 - 2x - 3 = 0$$

Then, $a = 1, b = -2,$ and $c = -3$
Sum of roots of the given equation,

$$\begin{aligned}(\alpha + \beta) &= \frac{-b}{a} \\ &= \frac{-(-2)}{1} = 2\end{aligned}$$

Product of roots,

$$\begin{aligned}(\alpha\beta) &= \frac{c}{a} \\ &= \frac{-3}{1} = -3\end{aligned}$$

Note that we must express the given function of α and β as functions of $(\alpha + \beta)$ and $\alpha\beta$ only.
But recall that;

$$\begin{aligned}(x + y)^2 &= x^2 + y^2 + 2xy \\ x^2 + y^2 &= (x + y)^2 - 2xy \\ x^3 + y^3 &= (x + y)(x^2 - xy + y^2) \\ x^3 + y^3 &= (x + y)[(x^2 + 2xy + y^2) - 3xy] \\ x^3 + y^3 &= (x + y)[(x + y)^2 - 3xy]\end{aligned}$$

By applying the above rules, we have that;

$$\begin{aligned}\alpha^2 + \beta^2 &= (\alpha + \beta)^2 - 2\alpha\beta \\ &= (2)^2 - 2(-3) \\ &= 4 + 6 = 10\end{aligned}$$

and

$$\begin{aligned}\alpha^3 + \beta^3 &= (\alpha + \beta)[(\alpha + \beta)^2 - 3\alpha\beta] \\ &= (2)[(2)^2 - 3(-3)] \\ &= 2(4 + 9) \\ &= 2(13) = 26\end{aligned}$$

(i) To find the equation whose roots are α^2, β^2;
Sum of roots $= \alpha^2 + \beta^2 = 10$
Product of roots $= \alpha^2\beta^2 = (\alpha\beta)^2 = (-3)^2 = 9$
Hence, the required equation is:

$$\begin{aligned} x^2 - \text{ (Sum of roots)}x + \text{ (Product of roots)} &= 0 \\ \therefore \qquad x^2 - 10x + 9 &= 0 \end{aligned}$$

(ii) To find the equation whose roots are α^3, β^3;
Sum of roots $= \alpha^3 + \beta^3 = 26$
Product of roots $= \alpha^3\beta^3 = (\alpha\beta)^3 = (-3)^3 = -27$
The required equation is:

$$\begin{aligned} x^2 - \text{ (sum of roots)}x + \text{ (product of roots)} &= 0 \\ x^2 - (\alpha^3 + \beta^3)x + (\alpha\beta)^3 &= 0 \\ x^2 - 26x - 27 &= 0 \end{aligned}$$

Example 2.2.10

If the equation $x^2 + px + 4 = 0$ has equal roots, find the possible values of p?

Solution

The equation $ax^2+bx+c = 0$ has equal roots if and only if $b^2 = 4ac$ or $b^2 - 4ac = 0$.
From the given equation, $a = 1,\ b = p,\ c = 4$;
Therefore, the above condition is satisfied if,

$$\begin{aligned} P^2 - 4(1)(4) &= 0 \quad \text{or} \quad p^2 - 16 = 0 \\ p^2 &= 16 \quad \text{or} \quad p = \pm\sqrt{16} \\ \therefore \qquad p &= \pm 4 \end{aligned}$$

Example 2.2.11

If one root of the equation $x^2 + tx + 2t = 0$ is 3 less than the other root, find the possible values of t.

Solution

Let α and β be the roots.
Further,let $\beta = \alpha - 3$.
Then, we have sum of roots as;

$$\alpha + (\alpha - 3) = 2\alpha - 3 = t \qquad (i)$$

Product of roots as;

$$\alpha(\alpha - 3) = 2t \qquad (ii)$$

By solving (i) and (ii), we observe that;

$$2\alpha - 3 = t \times 2 \quad \text{(i.e. multiplying equation (i) by 2 to get);}$$

$$4\alpha - 6 = 2t \qquad (iii)$$

By combining (ii) and (iii), we have;

$$\begin{aligned} \alpha(\alpha - 3) &= 4\alpha - 6 \\ \alpha^2 - 3\alpha - 4\alpha + 6 &= 0 \\ \alpha^2 - 7\alpha + 6 &= 0 \end{aligned}$$

Factorizing;

$$(\alpha - 6)(\alpha - 1) = 0$$

It is either $\alpha = 6$, or $\alpha = 1$
If we substitute for α in (i), we obtain;

$$\begin{aligned} 2(6) - 3 &= t \\ t &= 9 \end{aligned}$$

When $\alpha = 1$

$$\begin{aligned} 2(1) - 3 &= t \\ t &= 1 \end{aligned}$$

Therefore, the possible values of t are 9, and 1

Theorem 2.2.2. *To find the maximum or minimum value of the quadratic function $ax^2 + bx + c$, the real root x and $a \neq 0$.*

Proof.

$$ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$$

Now, by completing the square of the R.H.S.;

$$\Rightarrow \quad a\left[\left(x + \frac{b}{2a}\right)^2 + \frac{c}{a} - \frac{b^2}{4a^2}\right]$$

Since $\left(x + \frac{b}{2a}\right)^2$ is a complete square, it will have its minimum or maximum value when $x = \frac{-b}{2a}$. In this case, the value of the quadratic expression will be a minimum or maximum and their value will be;

$$y = a\left(\frac{c}{a} - \frac{b^2}{4a^2}\right) = \frac{4ac - b^2}{4a} \tag{2.2.4}$$

□

Remark 4 *This proof is true for negative or positive values of a.*

Corollary 2.2.1. *If x_1, and x_2 are the real roots, the quadratic equation,*

$$ax^2 + bx + c = 0; \quad a \neq 0$$

then the quadratic function, $y = ax^2 + bx + c$ will have its minimum or maximum value at;

$$x_m = \frac{x_1 + x_2}{2}, \quad \textit{and hence} \quad y_m = \frac{4ac - b^2}{4a}$$

Proof. Recall that from Theorem 2.2.2, the sum of roots x_1 and x_2 of a quadratic equation $ax^2 + bx + c = 0,\ a \neq 0$ is given as;

$$x_1 + x_2 = \frac{-b}{a}$$

Now, by substituting;

$$\begin{aligned} x_m &= \frac{x_1 + x_2}{2} \\ &= \frac{1}{2}\left(\frac{-b}{a}\right) \\ \therefore \quad x_m &= \frac{-b}{2a} \end{aligned}$$

Therefore, by substituting x_m for x in the quadratic function, we have the minimum or maximum value of y as;

$$\begin{aligned} y &= a\left(\frac{-b}{2a}\right)^2 + b\left(\frac{-b}{2a}\right) + c \\ &= a\left(\frac{b^2}{4a^2}\right) - \frac{b^2}{2a} + c \\ &= \frac{b^2}{4a} - \frac{b^2}{2a} + c \\ &= \frac{-b^2 + 4ac}{4a} \\ &= \frac{4ac - b^2}{4a} \\ \therefore \quad y &= \frac{4ac - b^2}{4a} \end{aligned}$$

□

Example 2.2.12

Find the minimum value of the curve or quadratic function:
$y = x^2 - 3x + 2$, and the corresponding value of x.

Solution

Comparing with standard form $ax^2 + bx + c = 0$; observe that $a = 1, b = -3$, and $c = 2$.
From the formula,

$$y = \frac{4ac - b^2}{4a}$$

On substitution, we have that;

$$\begin{aligned} \frac{4(1)(2) - (-3)^2}{4(1)} &= \frac{8 - 9}{4} \\ &= -\frac{1}{4} \end{aligned}$$

$\therefore$ the minimum value is $y = -\frac{1}{4}$
The value of x is therefore given as,

$$\begin{aligned} x &= \frac{-b}{2a} \\ &= \frac{-(-3)}{2(1)} = \frac{3}{2} \end{aligned}$$

Example 2.2.13

Find the maximum value of the quadratic function: $y = 1-2x-3x^2$, and hence the corresponding value of x.

Solution

From the equation, $a = -3$, $b = -2$ and $c = 1$, then;

$$\begin{aligned} y &= \frac{4ac - b^2}{4a} \\ &= \frac{4(-3)(1) - (-2)^2}{4(-3)} \\ &= \frac{-12 - 4}{-12} \\ &= \frac{-16}{-12} = \frac{4}{3} \end{aligned}$$

Hence the maximum value, $y = \frac{4}{3}$.
The value of x is obtained as;

$$\begin{aligned} x &= \frac{-b}{2a} \\ &= \frac{-(-2)}{2(-3)} = -\frac{1}{3} \end{aligned}$$

2.2.2 Special Quadratic Equations

Certain equations which do not appear to be of the quadratic types can often be reduced to this form by means of a substitution as in the following example.

Example 2.2.14

Solve the equation $x^4 - 3x^2 + 2 = 0$

Solution

Let $x^2 = z$. Thus, the given equation becomes a quadratic function in z and of the form:

$$z^2 - 3z + 2 = 0$$

which can now be solved using any of the methods as in previous examples in this section;

$$\begin{array}{rcl} (z-1)(z-2) & = & 0 \\ z-1 & = & 0 \quad \text{or} \quad z-2=0 \\ \therefore \qquad z & = & 1 \quad \text{or} \quad 2 \\ \text{but} \quad x^2 & = & z \\ \therefore \qquad x^2 & = & 1 \quad \text{or} \quad x^2 = 2 \\ x & = & \pm 1, \ \ \pm\sqrt{2} \end{array}$$

Example 2.2.15

Find the real roots of the equation,

$$x^2 + \frac{9}{x^2} - 4\left(x + \frac{3}{x}\right) - 6 = 0$$

Solution

Let $u = x + \frac{3}{x}$.
Then;

$$\begin{aligned} u^2 &= \left(x + \frac{3}{x}\right)^2 \\ &= x^2 + 6 + \left(\frac{3}{x}\right)^2 \\ &= x^2 + 6 + \frac{9}{x^2} \\ &= x^2 + \frac{9}{x^2} + 6 \\ \text{i.e.} \quad x^2 + \frac{9}{x^2} &= u^2 - 6 \end{aligned}$$

The given equation now becomes;

$$\begin{aligned} u^2 - 6 - 4u - 6 &= 0 \\ u^2 - 4u - 12 &= 0 \\ (u-6)(u+2) &= 0 \\ \therefore \quad u &= 6 \quad \text{or} \quad u = -2 \end{aligned}$$

But $u = x + \frac{3}{x}$
Then if $u = 6$, we have;

$$\begin{aligned} x + \frac{3}{x} &= 6 \\ x^2 + 3 &= 6x \\ x^2 - 6x + 3 &= 0 \\ (x-3)^2 + 3 - 3^2 &= 0 \end{aligned}$$

By completing the square or invoking the formula;

$$\begin{aligned} (x-3)^2 - 6 &= 0 \\ x - 3 &= \pm\sqrt{6} \\ \therefore \quad x &= 3 \pm \sqrt{6} \end{aligned}$$

Similarly, if $u = -2$
Then,

$$\begin{aligned} x + \frac{3}{x} &= -2 \\ x^2 + 3 &= -2x \\ \text{i.e.} \quad x^2 + 2x + 3 &= 0 \end{aligned}$$

By completing the square, we have;

$$\begin{aligned} (x+1)^2 + 3 - 1^2 &= 0 \\ (x+1)^2 + 2 &= 0 \\ (x+1)^2 &= -2 \\ x + 1 &= \pm\sqrt{-2} \\ \therefore \quad x &= -1 \pm \sqrt{-2} \end{aligned}$$

which gives complex roots $x = -1 \pm i\sqrt{2}$
$\therefore$ the real roots for $x = 3 \pm \sqrt{6}$

Example 2.2.16

Solve the equation,

$$z^4 + 2z^3 - z^2 + 2z + 1 = 0$$

Solution

This example comes under the general equation,

$$x^4 + ax^3 + bx^2 + ax + 1 = 0$$

which can be dealt with by first dividing through by z^2, giving;

$$\begin{aligned} z^2 + 2z - 1 + \frac{2}{z} + \frac{1}{z^2} &= 0 \\ z^2 + \frac{1}{z^2} + 2z + \frac{2}{z} - 1 &= 0 \\ z^2 + \frac{1}{z^2} + 2\left(z + \frac{1}{z}\right) - 1 &= 0 \end{aligned}$$

Now let $u = z + \frac{1}{z}$, then $u^2 = \left(z + \frac{1}{z}\right)^2$, on expanding, we have;

$$\begin{aligned} u^2 &= \left(z + \frac{1}{z}\right)^2 \\ &= z^2 + 2 + \frac{1}{z^2} \\ \text{or} \quad u^2 - 2 &= z^2 + \frac{1}{z^2} \end{aligned}$$

By substitution;

$$\begin{aligned} u^2 - 2 + 2u - 1 &= 0 \\ u^2 + 2u - 3 &= 0 \\ (u+3)(u-1) &= 0 \\ u = -3 \text{ or } \quad u &= 1 \end{aligned}$$

Now, when $u = -3$,

$$\begin{aligned} z + \frac{1}{z} &= -3 \\ z^2 + 1 &= -3z \\ z^2 + 3z + 1 &= 0 \end{aligned}$$

By completing the square;

$$\begin{aligned} \left(z + \frac{3}{2}\right)^2 + 1 - \left(\frac{3}{2}\right)^2 &= 0 \\ \left(z + \frac{3}{2}\right)^2 &= \frac{9}{4} - 1 \\ &= \frac{5}{4} \\ z + \frac{3}{2} &= \frac{\pm\sqrt{5}}{2} \\ \therefore \quad z &= \frac{-3}{2} \pm \frac{\sqrt{5}}{2} \\ &= \frac{-3 \pm \sqrt{5}}{2} \end{aligned}$$

Next when $u = 1$;

$$\begin{aligned} z + \frac{1}{z} &= 1 \\ z^2 + 1 &= z \\ \therefore \qquad z^2 - z + 1 &= 0 \end{aligned}$$

Completing the square;

$$\begin{aligned} \left(z - \frac{1}{2}\right)^2 + 1 - \frac{1}{4} &= 0 \\ \left(z - \frac{1}{2}\right)^2 &= \frac{-3}{4} \\ z - \frac{1}{2} &= \frac{\pm\sqrt{-3}}{2} \\ \therefore \qquad z &= \frac{1}{2} \pm \frac{\sqrt{-3}}{2} \\ &= \frac{1 \pm \sqrt{-3}}{2} \quad \text{(complex roots)} \end{aligned}$$

$\therefore$ the real root of $z = \frac{-3+\sqrt{5}}{2}, \quad \frac{-3-\sqrt{5}}{2}$

Example 2.2.17

Solve the equation,

$$\sqrt{(x-1)} + \sqrt{(x+4)} = \sqrt{(3x+10)},$$

where each term is real.

Solution

Squaring both sides of the given equation;

$$\begin{aligned} [\sqrt{(x-1)} + \sqrt{(x+4)}]^2 &= [\sqrt{3x+10)}]^2 \\ (\sqrt{(x-1)} + \sqrt{(x+4)})(\sqrt{(x-1)} + \sqrt{(x+4)}) &= 3x + 10 \\ (x-1) + 2\sqrt{((x-1)(x+4))} + (x+4) &= 3x + 10 \end{aligned}$$

$$\text{i.e.,} \quad 2\sqrt{(x-1)(x+4)} = 3x + 10 - 2x - 3$$
$$= x + 7$$

Squaring both sides again, we have;

$$\begin{aligned} 4(x-1)(x+4) &= (x+7)^2 \\ 4(x^2+3x-4) &= x^2+14x+49 \\ 4x^2+12x-16-x^2-14x-49 &= 0 \\ \therefore \quad 3x^2-2x-65 &= 0 \qquad (i) \end{aligned}$$

Which is now a quadratic equation in x.
By completing the square;

$$\begin{aligned} x^2 - \frac{2}{3}x &= \frac{65}{3} \\ \left(x - \frac{2}{6}\right)^2 &= \frac{65}{3} + \left(\frac{1}{3}\right)^2 \\ &= \frac{196}{9} \\ \therefore \quad x - \frac{1}{3} &= \pm\frac{14}{3} \\ &= \frac{1}{3} \pm \frac{14}{3} \\ x &= 5 \quad \text{or} \quad -\frac{13}{3} \end{aligned}$$

2.2.3 Simultaneous Quadratic Equations

A *homogeneous* function is a mathematical expression involving any number of variables all of which are of the same degree. For example, a homogeneous function of the second degree involving two variables x and y is one in which each term is of the second degree in x and y and is of the form

$$ax^2 + bxy + cy^2; \quad a \neq 0, \quad c \neq 0$$

where a, b, c are constants.

Consequently, a *simultaneous quadratic equation* is made of two homogeneous equations, one being linear and the other quadratic as

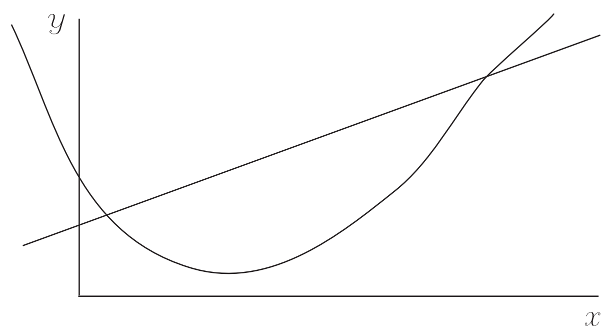

Figure 2.5: Graph of simultaneous quadratic equations

shown diagrammatically in Figure 2.5. In this section, two different types of simultaneous quadratic equations involving two variables will be considered.

(i) One linear and one quadratic equation.

(ii) Two homogeneous quadratic expressions, equal to a constant to form two equations.

Type (i)

This type is solved as follows: From the linear equation, find the value of one of the variables in terms of the other and substitute this in the quadratic equation, thus obtaining a quadratic equation in a single variable from which two values of that variable can be obtained. The corresponding values of the other variable is now obtained by using these values in the linear equation.

Example 2.2.18

Find the values of x and y which satisfy the equations:
$x^2 - 2xy + y^2 = 4$, and $2x - 3y = 5$.

Solution

$$\begin{aligned} x^2 - 2xy + y^2 &= 4 \qquad (i) \\ 2x - 3y &= 5 \qquad (ii) \end{aligned}$$

From (ii);

$$\begin{aligned} 2x &= 5+3y \\ \therefore \quad x &= \frac{5+3y}{2} \qquad (iii) \end{aligned}$$

Substituting (iii) into (i) yields;

$$\begin{aligned} \left(\frac{5+3y}{2}\right)^2 - 2y\left(\frac{5+3y}{2}\right) + y^2 &= 4 \\ (5+3y)^2 - 4y(5+3y) + 4y^2 &= 4(4) \\ 25 + 30y + 9y^2 - 20y - 12y^2 + 4y^2 &= 16 \\ y^2 + 10y + 9 &= 0 \qquad (iv) \end{aligned}$$

Factorizing (iv);

$$\begin{aligned} (y+9)(y+1) &= 0 \\ \therefore \quad y &= -9 \quad \text{or} \quad -1 \end{aligned}$$

By substituting for the values of y in (iii);
for $y = -9$;

$$x = \frac{5+3(-9)}{2} = -11$$

and for $y = -1$;

$$x = \frac{5+3(-1)}{2} = 1$$

$\therefore$ the values of $x = 1, y = -1$ and $x = -11, y = -9$

Example 2.2.19

Find the real values of x and y which satisfy the equations:
$2x^2 - xy - y^2 = 8$ and $xy = 6$.

Solution

$$\begin{aligned} 2x^2 - xy - y^2 &= 8 \qquad (i) \\ xy &= 6 \qquad (ii) \end{aligned}$$

From equation (ii);

$$y = \frac{6}{x} \qquad (iii)$$

Now substituting (iii) in (i), then;

$$\begin{aligned} 2x^2 - x\left(\frac{6}{x}\right) - \left(\frac{6}{x}\right)^2 &= 8 \\ 2x^4 - 6x^2 - 36 &= 8x^2 \\ 2x^4 - 6x^2 - 8x^2 - 36 &= 0 \\ 2x^4 - 14x^2 - 36 &= 0 \qquad (iv) \end{aligned}$$

Now, let $x^2 = u$, and equation (iv) becomes;

$$2u^2 - 14u - 36 = 0 \qquad (v)$$

which is now a quadratic equation in u;

$$\begin{aligned} u^2 - \frac{14}{2}u - \frac{36}{2} &= 0 \\ u^2 - 7u - 18 &= 0 \\ (u-9)(u+2) &= 0 \\ \therefore \quad u &= 9 \quad (or) \quad -2 \end{aligned}$$

But $x^2 = u$;
Thus, using $u = 9$; $\quad x^2 = 9$,

$$\therefore \quad x = \pm\sqrt{3}$$

and using $u = -2$; $\quad x^2 = -2$,

$$\therefore \quad x = \pm i\sqrt{2} \quad \text{(which is a complex root)}$$

Now, since the real value of $x = \pm 3$, obtain y from (iii) by substituting for x;
Thus, when $x = 3$;

$$y = \frac{6}{3} = 2$$

and using $x = -3$;

$$y = \frac{6}{-3} = -2$$

$\therefore$ the real values of x and y are:
$x = 3, y = 2$; and $x = -3, y = -2$

Type (ii)

Let the two homogeneous functions in x and y be,

$$a_0x^2 + a_1xy + a_2y^2$$

and

$$b_0x^2 + b_1xy + b_2y^2$$

where a_i's and b_i's are constants; $i = 0, 1, 2,$ and if the equations are;

$$\begin{aligned} a_0x^2 + a_1xy + a_2y^2 &= c_1 && (i) \\ b_0x^2 + b_1xy + b_2y^2 &= c_2 && (ii) \end{aligned}$$

Then, by multiplying equation (i) by c_2 and equation (ii) by c_1 and subtracting, an equation of the form;

$$Ax^2 + Bxy + Cy^2 = 0$$

is obtained, from which two values of one variable can be determined in terms of the other.

These values are now substituted in one of the equations (i) and (ii) each giving two equations in a single variable.

To find the corresponding values of the other variables these values are then substituted in the linear equations which were used to obtain the specific values.

Example 2.2.20

Solve the equations, $x^2 - xy + 3y^2 = 15$ and $3x^2 - 2y^2 = -5$.

Solution

$$\begin{aligned} x^2 - xy + 3y^2 &= 15 && (i) \\ 3x^2 - 2y^2 &= -5 && (ii) \end{aligned}$$

$(ii) \times 3$ gives;

$$9x^2 - 6y^2 = -15 \qquad (iii)$$

$(i) + (iii)$ gives;

$$10x^2 - xy - 3y^2 = 0 \qquad (iv)$$

By factorizing, we have;

$$(5x - 3y)(2x + y) = 0$$

Thus,

$$5x = 3y \quad \text{or} \quad 2x = -y$$

$$\begin{aligned} \therefore \quad x &= \frac{3y}{5} && (v) \\ \text{or} \quad x &= \frac{-y}{2} && (vi) \end{aligned}$$

Now substituting (v) into (ii), we have;

$$\begin{aligned} 3\left(\frac{3y}{5}\right)^2 - 2y^2 &= -5 \\ 27y^2 - 50y^2 &= -125 \\ -23y^2 &= -125 \\ y^2 &= \frac{125}{23} \\ \therefore \quad y &= \pm 5\sqrt{\frac{5}{23}} \end{aligned}$$

Now substituting for $y = \pm 5\sqrt{\frac{5}{23}}$ into (v);

$$\begin{aligned} x &= \frac{3}{5}\left(\pm 5\sqrt{\frac{5}{23}}\right) \\ &= \pm 3\sqrt{\frac{5}{23}} \end{aligned}$$

Similarly, substituting (vi) into (ii), we obtain;

$$\begin{aligned} 3\left(\frac{-y}{2}\right)^2 - 2y^2 &= -5 \\ 3y^2 - 8y^2 &= -20 \\ -5y^2 &= -20 \\ y^2 &= 4 \\ \therefore \quad y &= \pm 2 \end{aligned}$$

Now, substituting $y = \pm 2$ into (vi), we get;

$$x = -\frac{(\pm 2)}{2} = \mp\frac{2}{2} = \pm 1$$

$\therefore$ the solutions are:
$x = \pm 3\sqrt{\frac{5}{23}}, y = \pm 5\sqrt{\frac{5}{23}}$ and $x = \pm 1, y = \pm 2$.

Example 2.2.21

Find all the values of x and y which satisfy the simultaneous equations: $x^2 - 2xy - y^2 = 14, \quad 2x^2 + 3xy + y^2 = -2$

Solution

$$\begin{aligned} x^2 - 2xy - y^2 &= 14 & (i) \\ 2x^2 + 3xy + y^2 &= -2 & (ii) \end{aligned}$$

$(i) + (ii)$ gives;

$$3x^2 + xy = 12 \qquad (iii)$$

$(ii) \times 6$ gives;

$$12x^2 + 18xy + 6y^2 = -12 \qquad (iv)$$

$(iii) + (iv)$ gives;

$$15x^2 + 19xy + 6y^2 = 0 \qquad (v)$$

Factorizing (v);

$$(5x + 3y)(3x + 2y) = 0$$

Thus,

$$5x = -3y \quad \text{or} \quad 3x = -2y$$

$$\therefore \quad x = \frac{-3}{5}y \qquad (vi)$$

$$\text{or} \quad x = -\frac{2}{3}y \qquad (vii)$$

Substituting (vi) into (iii);

$$\begin{aligned}
3\left(\frac{-3}{5}y\right)^2 + \left(\frac{-3}{5}y\right)y &= 12 \\
27y^2 - 15y^2 &= 300 \\
12y^2 &= 300 \\
y^2 &= 25 \\
y &= \pm 5
\end{aligned}$$

Next, substituting $y = \pm 5$ in (vi);

$$x = \frac{-3}{5}(\pm 5) = \frac{\pm 15}{5} = \pm 3$$

Similarly, substituting (vii) into (iii), yields;

$$\begin{aligned}
3\left(\frac{-2}{3}y\right)^2 + \left(\frac{-2}{3}y\right)y &= 12 \\
12y^2 - 6y^2 &= 108 \\
6y^2 &= 108 \\
y^2 &= 18 \\
\therefore \quad y &= \pm 3\sqrt{2}
\end{aligned}$$

Then, substituting $y = \pm 3\sqrt{2}$ into (vii),

$$\begin{aligned}
x &= -\frac{2}{3}(\pm 3\sqrt{2}) \\
&= \frac{\mp 6\sqrt{2}}{3} \\
&= \pm 2\sqrt{2}
\end{aligned}$$

$\therefore$ the values of x and y are;

$$x = \pm 3, y = \pm 5 \quad \text{and} \quad x = \pm 2\sqrt{2}, y = \pm 3\sqrt{2}$$

.

Further Examples

Example 2.2.22

Determine the real solutions of the simultaneous equations,
$x^3 + y^3 = 35$, and $x^2y + xy^2 = 30$

Solution

$$\begin{aligned} x^3 + y^3 &= 35 & (i) \\ x^2y + xy^2 &= 30 & (ii) \end{aligned}$$

Factorizing (i) and (ii) yields;

$$\begin{aligned} (x+y)(x^2 - xy + y^2) &= 35 & (iii) \\ xy(x+y) &= 30 & (iv) \end{aligned}$$

Observe that the factor, $(x + y)$ is common to both (iii) and (iv). Thus, dividing (iii) by (iv);

$$\begin{aligned} \frac{x^2 - xy + y^2}{xy} &= \frac{35}{30} \\ &= \frac{7}{6} \end{aligned}$$

Hence,

$$\begin{aligned} 6(x^2 - xy + y^2) &= 7xy \\ 6x^2 - 6xy - 7xy + 6y^2 &= 0 \\ 6x^2 - 13xy + 6y^2 &= 0 \end{aligned}$$

Factorizing;

$$\begin{aligned} (3x - 2y)(2x - 3y) &= 0 \\ 3x &= 2y \quad \text{or} \quad 2x = 3y \end{aligned}$$

Thus,

$$\begin{aligned} x &= \frac{2}{3}y & (v) \\ \text{or} \quad x &= \frac{3}{2}y & (vi) \end{aligned}$$

Now substituting (v) into (i), we obtain;

$$\begin{aligned}\left(\frac{2}{3}y\right)^3 + y^3 &= 35 \\ 8y^3 + 27y^3 &= 945 \\ 35y^3 &= 945 \\ y^3 &= 27 \\ \therefore \quad y &= 3\end{aligned}$$

Putting $y = 3$ into (v);

$$x = \frac{2}{3}(3) = 2$$

Similarly, by substituting (vi) into (i);

$$\begin{aligned}\left(\frac{3}{2}y\right)^3 + y^3 &= 35 \\ 27y^3 + 8y^3 &= 280 \\ 35y^3 &= 280 \\ y^3 &= 8 \\ \therefore \quad y &= 2\end{aligned}$$

Putting $y = 2$ into (vi),

$$x = \frac{3}{2}(2) = 3.$$

$\therefore$ the values $x = 3, y = 2$, and $x = 2, y = 3$

Example 2.2.23

Show that the simultaneous equation:
$xy - x - 2y + 2 = 3$, $2xy - 5x + 4y - 10 = 15$ has real solutions.

Solution

$$\begin{aligned}xy - x - 2y + 2 &= 3 \qquad (i) \\ 2xy - 5x + 4y - 10 &= 15 \qquad (ii)\end{aligned}$$

Factorizing (i) and (ii);

$$\begin{aligned} (x-2)(y-1) &= 3 && (iii) \\ (x+2)(2y-5) &= 15 && (iv) \end{aligned}$$

From equation (iv);

$$x+2 = \frac{15}{2y-5} \qquad (v)$$

Since the L.H.S. of equation (iii) has a factor of $(x-2)$, then we can subtract the number 4 from both sides of equation (v) to get;

$$x-2 = \frac{15}{2y-5} - 4 \qquad (vi)$$

Now, substituting (vi) into $(x-2)$ in (iii), we obtain;

$$\begin{aligned} \left(\frac{15}{2y-5} - 4\right)(y-1) &= 3 \\ 15y - 15 - 4y(2y-5) + 4(2y-5) &= 3(2y-5) \\ 15y - 15 - 8y^2 + 20y + 8y - 20 - 6y + 15 &= 0 \\ -8y^2 + 37y - 20 &= 0 \\ 8y^2 - 37y + 20 &= 0 \end{aligned}$$

Factorizing;

$$\begin{aligned} (y-4)(8y-5) &= 0 \\ y &= 4 \quad \text{or} \quad 8y = 5 \\ \therefore \quad y &= 4 \quad \text{or} \quad \frac{5}{8} \end{aligned}$$

By using $y = 4$ in (v);

$$\begin{aligned} x+2 &= \frac{15}{8-5} \\ &= 5 \\ \therefore \quad x &= 3 \end{aligned}$$

Next, putting $y = \frac{5}{8}$ in (v) gives;

$$\begin{aligned} x + 2 &= \frac{15}{2(\frac{5}{8}) - 5} \\ &= -4 \\ \therefore \qquad x &= -6 \end{aligned}$$

$\therefore$ the real solutions are; $x = 3$ and $y = 4$ or $x = -6$ and $y = \frac{5}{8}$

Chapter 3

Identities, Remainder Theorem and Factor Theorem

In the previous chapter, we discussed the theory of quadratic equations and its solutions. However, when the degree of the variables in the equation becomes larger than 2, the solutions of the polynomial equation using quadratic methods fails. It is therefore the inadequacy of quadratic methods to solving polynomial equations of degree greater than 2 that led to the subject of this chapter.

This chapter derives its principles from the basic laws of number systems guiding the representation of rational numbers (fractions) using the arithmetic operator – division. The introduction of identities here is to relate upon simplification, the rational function on the LHS to their factors (quotients) and remainder on the RHS of the equation.

3.1 Identities

Definition 1 *A conditional equation is an equation that is true for only finite number of values of the variable present in it.*

For example, $\frac{1}{2}x + \frac{2}{3}x = \frac{1}{3}$ is a conditional equation since it is true by the value $x = \frac{2}{7}$ only, similarly the quadratic equation,

$x^2 - 4x + 3 = 0$ is a conditional equation since it is true for only two values of $x = 1$ and $x = 3$. However, when an equation is identical, it simply implies that both the L.H.S. and the R.H.S. of the equation are the same upon simplification for any values of the variables.

An *identity* is therefore an equation that is true for all values of the variable (or variables) contained in it (or, when on simplification, both sides are identical). For example, $(x-1)^2 = x^2 - 2x + 1$ is true for all values of x, hence it is an identity. Customarily, it is written:

$$(x-1)^2 \equiv x^2 - 2x + 1$$

. Similarly, $(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$ for all values of x and y can be written in the form:

$$(x+y)^3 \equiv x^3 + 3x^2y + 3xy^2 + y^3$$

being an identity.

3.2 Remainder Theorem

Let $f(x)$ be a polynomial function of degree greater than 1, the theorem states that the remainder on dividing $f(x)$ by a linear function $ax + b$ say, $f(-b/a)$, where a, b are constants and $ax + b \neq 0$. When $f(x)$ is divided by $ax+b$, let the quotient be $\varphi(x)$ and the remainder R. Then, it is known by algebraic rules that R will be a constant and will not contain x, so that we write,

$$f(x) \equiv (ax + b)\varphi(x) + R \tag{3.2.1}$$

(this is an identity valid for all values of x).

Now, substituting for x in this identity, i.e. $x = -\frac{b}{a}$,

$$\begin{aligned} f\left(-\frac{b}{a}\right) &= 0 \times \varphi\left(-\frac{b}{a}\right) + R \\ &= 0 + R \\ \text{i.e.,} \quad R &= f\left(-\frac{b}{a}\right) \end{aligned}$$

Suppose, we put $a = 1$ and b is replaced by $(-\beta)$, then the result becomes;

$$R = f(\beta),$$

i.e., when $f(x)$ is divided by $(x - \beta)$, the remainder has the value $f(\beta)$.

Example 3.2.1

Find the remainder on dividing $2x^3 - x^2 - 5x + 1$ by (i) $2x - 5$ (ii) $x + 3$.

Solution

Let $f(x) = 2x^3 - x^2 - 5x + 1$

(i) Divisor: $2x - 5$, the required remainder;

$$\begin{aligned} R &= f(\frac{5}{2}) \\ &= 2\left(\frac{5}{2}\right)^3 - \left(\frac{5}{2}\right)^2 - 5\left(\frac{5}{2}\right) + 1 \\ &= \frac{125}{4} - \frac{25}{4} - \frac{25}{2} + 1 \\ \therefore \quad R &= 13\frac{1}{2} \end{aligned}$$

(ii) Divisor: $x + 3$, the required remainder;

$$\begin{aligned} R &= f(-3) \\ &= 2(-3)^3 - (-3)^2 - 5(-3) + 1 \\ &= -54 - 9 + 15 + 1 \\ \therefore \quad R &= -47 \end{aligned}$$

3.3 Factor Theorem

In a polynomial function, finding the remainders is of less importance, except when the remainder is zero. Observe that in the previous worked example, the divisor would be a factor of the polynomial if and only if the remainder is zero ($R = 0$). Thus, the remainder theorem can be used to factorize polynomials. When used in this way, it is sometimes referred to as the factor theorem *which states that, if* $(ax+b)$ *is a factor of the polynomial* $f(x)$*, then* $f\left(-\frac{b}{a}\right) = 0$*, and conversely, if* $f(-\frac{b}{a}) = 0$*, then* $(ax+b)$ *is a factor of* $f(x)$*.*

Now, suppose $f(x)$ has $(ax+b)$ as a factor, then the remainder R obtained by dividing $f(x)$ by $(ax+b)$ must be zero, and it has been established that

$$R = f(-\frac{b}{a}), \quad \text{then} \quad f(-\frac{b}{a}) = 0$$

If $(ax+b)$ is a factor of $f(x)$ and $f\left(-\frac{b}{a}\right) = 0$, then $R = 0$ and $(ax+b)$ must be a factor of $f(x)$.

Consequently, if we put $\alpha = -\frac{b}{a}$, then $(x-\alpha)$ is a factor of $f(x)$, so that $f(\alpha) = 0$; and if $f(\alpha) = 0$, then $(x-\alpha)$ is a factor of $f(x)$.

Remark 5 *It is noteworthy that finding the values of* x *that makes* $f(x) = 0$ *(i.e. the roots of a given equation) is obtained by trial and error using the factor theorem. In other words, if one root of an equation say,* $x = \alpha$ *be found by this method, then* $f(x)$ *can be divided by the known factor* $(x-\alpha)$ *and the remaining roots can be obtained by equating the polynomial quotient obtained to zero and continuing in this manner until a quadratic quotient remains whose roots can be obtained by elementary factorization or using the same method to obtain the last two roots.*

Example 3.3.1

Using the remainder theorem, factorize as fully as possible,

(i) $x^3 + 6x^2 + 11x + 6$,

(ii) $4x^3 + x^2 - 27x + 18$.

Solution

(i) Considering the constant term 6, the only possible factors are $\pm1, \pm2, \pm3, \pm6$, and hence the only possible factors of the given function $f(x)$ are, $(x \pm 1)$, $(x \pm 2)$, $(x \pm 3)$, $(x \pm 6)$.

Starting with the lowest values;
Let,

$$\begin{aligned} f(x) &\equiv x^3 + 6x^2 + 11x + 6 \\ \text{i.e.} \quad f(-1) &= (-1)^3 + 6(-1)^2 + 11(-1) + 6 = 0 \end{aligned}$$

$\therefore \quad (x+1)$ is a factor of $f(x)$
Now, divide $f(x)$ by $x + 1$;

$$\begin{array}{r|l} & x^2 + 5x + 6 \\ \hline x+1 & x^3 + 6x^2 + 11x + 6 \\ & x^3 + x^2 \\ \hline & \quad 5x^2 + 11x \\ & \quad 5x^2 + 5x \\ \hline & \qquad 6x + 6 \\ & \qquad 6x + 6 \\ \hline \end{array}$$

$$\begin{aligned} \therefore \quad f(x) &= (x+1)(x^2 + 5x + 6) \\ &= (x+1)(x+2)(x+3) \end{aligned}$$

(ii) Similarly, if we divide through by the coefficient of x^3, the constant term becomes $\frac{18}{4}$ whose possible factors are $\pm\frac{1}{4}$, $\pm\frac{3}{4}$, ± 1, ± 2, ± 3, $\pm\frac{18}{4}$ and hence the only possible factors of $f(x)$ are: $\left(x \pm \frac{1}{4}\right)$, $\left(x \pm \frac{3}{4}\right)$, $(x \pm 1)$, $(x \pm 2)$, $(x \pm 3)$, $\left(x \pm \frac{18}{4}\right)$
Let $f(x) \equiv 4x^3 + x^2 - 27x + 18$
Thus,

$$f\left(-\frac{1}{4}\right) = 4\left(-\frac{1}{4}\right)^3 + \left(-\frac{1}{4}\right)^2 - 27\left(-\frac{1}{4}\right) + 18 \neq 0$$

Similarly, trying $\frac{1}{4}$, we have;

$$f\left(\frac{1}{4}\right) = 4\left(\frac{1}{4}\right)^3 + \left(\frac{1}{4}\right)^2 - 27\left(\frac{1}{4}\right) + 18 \neq 0$$

Therefore, from the two substitutions above; $\left(x \pm \frac{1}{4}\right)$ is not a factor of $f(x)$.
Now, let us try $-\frac{3}{4}$;

$$f\left(-\frac{3}{4}\right) = 4\left(-\frac{3}{4}\right)^3 + \left(-\frac{3}{4}\right)^2 - 27\left(-\frac{3}{4}\right) + 18 \neq 0$$

and trying $\frac{3}{4}$;

$$f\left(\frac{3}{4}\right) = 4\left(\frac{3}{4}\right)^3 + \left(\frac{3}{4}\right)^2 - 27\left(\frac{3}{4}\right) + 18 = 0$$

$\therefore \quad \left(x - \frac{3}{4}\right)$ or $(4x - 3)$ is a factor of $f(x)$.
Dividing through by $4x - 3$ we obtain;

$$\begin{array}{r|llll}
 & x^2 & + \; x & - \; 6 & \\
\hline
4x - 3 & 4x^3 & + \; x^2 & - \; 27x + 18 & \\
 & 4x^3 & - \; 3x^2 & & \\
\hline
 & & 4x^2 & - \; 27x & \\
 & & 4x^2 & - \; 3x & \\
\hline
 & & & - \; 24x + 18 & \\
 & & & - \; 24x + 18 & \\
\hline
\end{array}$$

$$\begin{aligned} \therefore \quad f(x) &= (4x-3)(x^2+x-6) \\ &= (4x-3)(x-2)(x+3) \end{aligned}$$

Example 3.3.2

$(x - 2)$ is a factor of $2x^3 + \alpha x^2 + \beta x - 2$, and when this expression is divided by $(x + 3)$ the remainder is -50.

(i) Find the values of the constants α and β.

(ii) With these values, factorize the expression completely.

(iii) Hence solve the equation $2x^3 + \alpha x^2 + \beta x - 2 = 0$.

Solution

Let $f(x) \equiv 2x^3 + \alpha x^2 + \beta x - 2$

(i) If $x - 2$ is a factor of $f(x)$, then $f(2) = 0$, so that,

$$\begin{aligned} f(2) = 2(2)^3 + \alpha(2)^2 + \beta(2) - 2 &= 0 \\ \therefore \quad 16 + 4\alpha + 2\beta - 2 &= 0 \\ \text{or} \quad 4\alpha + 2\beta &= -14 \qquad (i) \end{aligned}$$

The remainder, when $f(x)$ is divided by $(x+3)$, is $f(-3)$ and this equals -50.
i.e.,

$$\begin{aligned} f(-3) = 2(-3)^3 + \alpha(-3)^2 + \beta(-3) - 2 &= -50 \\ \therefore \quad -54 + 9\alpha - 3\beta - 2 &= -50 \\ \text{or} \quad 9\alpha - 3\beta &= 6 \qquad (ii) \end{aligned}$$

Now, solving (i) and (ii) simultaneously, we obtain $\alpha = -1$ and $\beta = -5$.
Hence $f(x) = 2x^3 - x^2 - 5x - 2$.

(ii) Now divide $f(x)$ by $(x - 2)$;

$$\begin{array}{r|llllll} & 2x^2 & + & 3x & + & 1 \\ \hline x - 2 & 2x^3 & - & x^2 & - & 5x - 2 \\ & 2x^3 & - & 4x^2 & & \\ \hline & & & 3x^2 & - & 5x \\ & & & 3x^2 & - & 6x \\ \hline & & & & & x - 2 \\ & & & & & x - 2 \\ \hline \end{array}$$

$$\begin{aligned} \therefore \quad f(x) &= (x-2)(2x^2 + 3x + 1) \\ &= (x-2)(2x+1)(x+1) \end{aligned}$$

(iii) The equation $2x^3 + \alpha x^3 + \beta x - 2 = 0$ implies that;

$$f(x) = 2x^3 - x^2 - 5x - 2 = 0$$

Hence, the roots of the equation are $x = 2$, $-\frac{1}{2}$ and -1.

3.4 Partial Fractions

Consider the rational function $P_m(x)/Q_n(x)$, where $P_m(x)$ and $Q_n(x)$ are polynomials of degree m and n respectively, this fraction can be expressed as the sum of two or more simpler fractions according to certain definite rules and the resolved components resulting from the break down are called *partial fractions*, usually referred to as the fractional equivalence of the given function.

If $m \geq n$, (i.e. when the degree of the numerator is greater than or equal to the degree of the denominator) then the numerator $P_m(x)$ is divided by the denominator $Q_n(x)$ to get a quotient and a remainder. Thus,

$$\frac{P_m(x)}{Q_n(x)} = G_s(x) + \frac{R_t(x)}{Q_n(x)}, \quad (t < n) \tag{3.4.1}$$

where $R_t(x)$ is the remainder and is of degree less than the degree of $Q_n(x)$ $(ie.t < n)$. We then proceed to resolve $R_t(x)/Q_n(x)$ into partial fractions.

We shall use examples to illustrate the various cases that may arise, but for clarity, we shall categorize these various cases as follows;

Case I

If $Q_n(x)$ can be expressed as a product of linear factors, then to every linear factor of the form $(a_i x + b_i)$ of $Q_n(x)(i = 1, 2, ..., n)$ there will be a corresponding partial fraction i.e.,

$$\frac{P_m(x)}{Q_n(x)} \equiv \frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + ... + \frac{A_n}{(ax+b)^n}, m < n$$

Case II

If $Q_n(x)$ has repeated factors, then to every repeated factor of the form $(ax+b)^i; (i = 1, 2, ..., n)$ of $Q_n(x)$ there will be n corresponding partial fractions i.e.,

$$\frac{P_m(x)}{Q_n(x)} \equiv \frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + ... + \frac{A_n}{(ax+b)^n}; m < n$$

Case III

If $Q_n(x)$ has irreducible or quadratic factors, then to every quadratic factor of the form $a_ix^2 + b_ix + c_i,\ (i = 1, 2, ..., n) of Q_{2n}(x)$ there will be a corresponding partial fraction of the form;

$$\frac{P_m(x)}{Q_{2n}(x)} \equiv \frac{A_ox + A_1}{a_1x^2 + b_1x + c_1} + \frac{A_2x + A_3}{a_2x^2 + b_2x + c_2} + ...$$

$$... + \frac{A_nx + A_{n+1}}{a_nx^2 + b_nx + c_n}; \quad m < 2n$$

Case IV

If $Q(x)$ has repeated quadratic factors, then to every repeated quadratic factor of the form $(ax^2 + bx + c)^i,\ (i = 1, 2, 3, ..., n)$ of $Q_{2n}(x)$, there will be corresponding partial fractions of the form;

$$\frac{P_m(x)}{Q_{2n}(x)} \equiv \frac{A_ox + A_1}{(ax^2 + bx + c)} + \frac{A_2x + A_3}{(ax^2 + bx + c)^2} + ... + \frac{A_nx + A_{n+1}}{(ax^2 + bx + c)^n},$$

$m < 2n$

Remark 6 *To determine the constants $A_0, A_1, A_2, ..., A_{n+1}$, the whole identity written down is multiplied throughout by $Q_n(x)$ given rise to a second identity, from which the values of the constants are obtained. Values of x that make each linear or quadratic factor of $Q_n(x)$are substituted (this simplifies the working), and by equating coefficients of like terms, all the required constants are then determined.*

Case I: Example 3.4.1

Express the following rational functions in partial fractions:

$$(i)\ \frac{x}{x^2 - 4} \quad (ii)\ \frac{x^2 + 6}{x^2 + 3x - 4} \quad (iii)\ \frac{3x + 1}{(x + 2)(x - 1)(x + 3)}$$

Solution

(i)

$$\frac{x}{x^2-4} \equiv \frac{x}{(x-2)(x+2)}$$

Set

$$\frac{x}{(x-2)(x+2)} \equiv \frac{A}{x-2} + \frac{B}{x+2}$$

Then, by multiplying through by $(x-2)(x+2)$;

$$x \equiv A(x+2) + B(x-2)$$

But, since the above equation is an identity, it is true for all values of x;
i.e. $x = Ax + 2A + Bx - 2B$
By equating coefficients, we have;

$$\begin{aligned} Ax + Bx &= x \\ (A+B)x &= x \\ \therefore \quad A + B &= 1 \qquad (i) \end{aligned}$$

Similarly,

$$2A - 2B = 0 \qquad (ii)$$

Solving equations (i) and (ii) we have; $A = \frac{1}{2},\ B = \frac{1}{2}$

$$\begin{aligned} \therefore \quad \frac{x}{x^2-4} &= \frac{\frac{1}{2}}{x-2} + \frac{\frac{1}{2}}{x+2} \\ &= \frac{1}{2(x-2)} + \frac{1}{2(x+2)} \end{aligned}$$

(ii) From

$$\frac{x^2+6}{x^2+3x-4}$$

the degree of numerator $= 2$ and the degree of denominator $= 2$. This indicates that the denominator and the numerator have the same degree. But recall that one of the necessary condition for resolving into partial fraction is that the degree of the numerator $P_m(x)$ must be less than the degree of the denominator $Q_n(x)$, i.e.,$m < n$. Therefore we divide before resolving the remainder into partial fractions;

$$
\begin{array}{r|lll}
 & 1 & & \\
\hline
x^2+3x-4 & x^2 & & +6 \\
 & x^2 & +3x-4 & \\
\hline
 & & -3x+10 &
\end{array}
$$

$$
\begin{aligned}
\therefore \quad \frac{x^2+6}{x^2+3x-4} &\equiv 1+\frac{-3x+10}{x^2+3x-4} \\
\text{i.e.} \quad \frac{10-3x}{x^2+3x-4} &\equiv \frac{10-3x}{(x+4)(x-1)} \\
\frac{10-3x}{(x+4)(x-1)} &\equiv \frac{A}{x+4}+\frac{B}{x-1}
\end{aligned}
$$

Multiplying through by $(x+4)(x-1)$;

$$
\begin{aligned}
10-3x &= A(x-1)+B(x+4) \\
10-3x &= Ax-A+Bx+4B.
\end{aligned}
$$

Now, equating the coefficients;

$$
\begin{aligned}
Ax+Bx &= -3x \\
(A+B)x &= -3x \\
A+B &= -3 \qquad (i)
\end{aligned}
$$

Similarly,

$$-A+4B = 10 \qquad (ii)$$

Solving equations (i) and (ii);

$$B=\frac{7}{5} \quad \text{and} \quad A=-\frac{22}{5}$$

$$\therefore \quad \frac{10-3x}{(x+4)(x-1)} = -\frac{22}{5(x+4)}+\frac{7}{5(x-1)}$$

Thus,

$$\frac{x^2+6}{x^2+3x-4} = 1-\frac{22}{5(x+4)}+\frac{7}{5(x-1)}$$

(iii) From

$$\frac{3x+1}{(x+2)(x-1)(x+3)}$$

the degree of numerator is less than the degree of denominator. So, set

$$\frac{3x+1}{(x+2)(x-1)(x+3)} \equiv \frac{A}{x+2} + \frac{B}{x-1} + \frac{C}{x+3}$$

By multiplying through by $(x+2)(x-1)(x+3)$;

$$3x+1 \equiv A(x-1)(x+3) + B(x+2)(x+3) + C(x+2)(x-1)$$

Observe here that by opening up the brackets, and equating the resultant coefficients as in equation (i) and (ii) above, would be rather ineffective, instead, we use the values of x that would make each linear factor of the denominator vanish, such that, we have for $x = 1$, $\quad 4 = B(3)(4) = 12B$, So that $B = \frac{1}{3}$

Similarly, for $x = -2$; $-5 = A(-3)(1) = -3A$, so that $A = \frac{5}{3}$ and for $x = -3$; $\quad -8 = C(-1)(-4) = 4C$, so that $C = -2$.

$$\therefore \qquad \frac{3x+1}{(x+2)(x-1)(x+3)} = \frac{5}{3(x+2)} + \frac{1}{3(x-1)} - \frac{2}{x+3}$$

Case II: Example 3.4.2

Express in partial fractions;

$$\text{(i)} \ \frac{x^4+2x^3-x+2}{(x-2)(x+1)^2}, \quad \text{(ii)} \ \frac{2x+3}{(2x-1)(x+1)^3}$$

Solution

(i) Since the degree of numerator is greater than the degree of the denominator , we divide before resolving the remainder into partial fractions;

$$\begin{aligned}(x-2)(x+1)^2 &= x^3+2x^2+x-2x^2-4x-2 \\ &= x^3-3x-2\end{aligned}$$

$$
\begin{array}{r|l}
 & x + 2 \\
\hline
x^3 - 3x - 2 & x^4 + 2x^3 \qquad\quad - x + 2 \\
 & x^4 \qquad\quad -3x^2 - 2x \\
\cline{2-2}
 & \quad 2x^3 + 3x^2 + x \\
 & \quad 2x^3 \qquad\quad - 6x - 4 \\
\cline{2-2}
 & \qquad\quad 3x^2 + 7x + 6
\end{array}
$$

$$\therefore \qquad \frac{x^4 + 2x^3 - x + 2}{(x-2)(x+1)^2} \quad = \quad x + 2 + \frac{3x^2 + 7x + 6}{(x-2)(x+1)^2}$$

Now, set

$$\frac{3x^2 + 7x + 6}{(x-2)(x+1)^2} \quad \equiv \quad \frac{A}{x-2} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

By multiplying through by $(x-2)(x+1)^2$ (i.e. L.C.M.);

$$3x^2 + 7x + 6 = A(x+1)^2 + B(x-2)(x+1) + C(x-2)$$

Now, use the values of x that make each linear factor of the denominator vanish, such that;
For $x = 2$,

$$3(2)^2 + 7(2) + 6 = A(3)^2$$

So that

$$A = \frac{32}{9} \quad \text{and hence} \quad C = -\frac{2}{3}$$

Furthermore, by equating the constants to obtain the value of B, we have;

$$
\begin{aligned}
6 &= A - 2B - 2C \\
\Rightarrow \quad 6 &= \frac{32}{9} - 2B - 2\left(-\frac{2}{3}\right) \\
&= \frac{32}{9} - 2B + \frac{4}{3} \\
2B &= -6 + \frac{32}{9} + \frac{4}{3} \\
18B &= -10 \\
\therefore \quad B &= -\frac{10}{18} = -\frac{5}{9}
\end{aligned}
$$

$$\therefore \quad \frac{3x^2+7x+6}{(x-2)(x+1)^2} = \frac{32}{9(x-2)} - \frac{5}{9(x+1)} - \frac{2}{3(x+1)^2}$$

Thus,

$$\frac{x^4+2x^3-x+2}{(x-2)(x+1)^2} = x+2+\frac{32}{9(x+2)} - \frac{5}{9(x+1)} - \frac{2}{3(x+1)^2}$$

(ii) Let

$$\frac{2x+3}{(2x-1)(x+1)^3} \equiv \frac{A}{2x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2} + \frac{D}{(x+1)^3}$$

Multiplying both sides by $(2x-1)(x+1)^3$,

$$\begin{aligned} 2x+3 &= A(x+1)^3 + B(2x-1)(x+1)^2 + C(2x-1)(x+1) \\ &\quad +D(2x-1) \end{aligned}$$

Suppose $x = \frac{1}{2}$,then;

$$\begin{aligned} 2\left(\frac{1}{2}\right)+3 &= A(\frac{3}{2})^3 + B(0) + 0 + 0 \\ 4 &= \frac{27}{8}A \\ \therefore \quad A &= \frac{32}{27} \end{aligned}$$

Similarly, if we put $x = -1$,

$$2(-1)+3 = D(-3) \quad \text{i.e.} \quad 1 = -3D \quad \text{or} \quad D = -\frac{1}{3}$$

Equating the constant terms;

$$\begin{aligned} 3 &= A(1)^3 + B(-1)(1)^2 + C(-1)(1) + D(-1) \\ 3 &= A - B - C - D \end{aligned}$$

Thus,

$$\begin{aligned} A - B - C - D &= 3 \\ \text{i.e.} \quad \frac{32}{27} - B - C + \frac{1}{3} &= 3 \\ \therefore \quad 27B + 27C &= -40 \qquad (i) \end{aligned}$$

Similarly, put $x = 1$;

$$\begin{aligned} 2(1) + 3 &= A(2)^3 + B(2)^2 + C(2) + D(1) \\ 5 &= 8A + 4B + 2C + D \\ 5 &= 8\left(\frac{32}{27}\right) + 4B + 2C + \left(-\frac{1}{3}\right) \\ \text{or} \quad 108B + 54C &= -112 \qquad (ii) \end{aligned}$$

Solving (i) and (ii) simultaneously, we obtain;

$$B = -\frac{16}{27} \quad \text{and} \quad C = -\frac{8}{9}$$

$$\therefore \quad \frac{2x+3}{(2x-1)(x+1)^3} = \frac{32}{27(2x-1)} - \frac{16}{27(x+1)} - \frac{24}{27(x+1)^2} - \frac{1}{3(x+1)^3}$$

Case III: Example 3.4.3

Express in partial fractions,

$$(i)\ \frac{8}{(x+1)(x^2+2)}, \quad (ii)\ \frac{x^6-1}{(x+1)(x^4-2x)}$$

.

Solution

(i) Observe that the numerator of the rational function corresponding to $x^2 + 2$ is a quadratic factor since it is irreducible, hence applying the rule of Case III, we have;

$$\frac{8}{(x+1)(x^2+2)} \equiv \frac{A}{x+1} + \frac{Bx+C}{x^2+2}$$

Multiplying both sides by $(x+1)(x^2+2)$;

$$\begin{aligned} 8 &\equiv A(x^2+2) + (Bx+C)(x+1) \\ \text{i.e. } 8 &\equiv A(x^2+2) + Bx(x+1) + C(x+1) \end{aligned}$$

If $x = -1$, $\quad 8 = A(3) + 0 = 3A \Rightarrow A = \frac{8}{3}$
If $x = 0$,

$$\begin{aligned} 8 &= A(2) + C(1) \\ &= 2A + C \\ 8 &= 2(\frac{8}{3}) + C \quad \text{or} \quad C = \frac{8}{3} \end{aligned}$$

Put $x = 1$ (say),

$$\begin{aligned} 8 &= A(3) + B(2) + C(2) \\ 8 &= 3A + 2B + 2C \\ \text{i.e.} \quad 8 &= 3(\frac{8}{3}) + 2B + 2(\frac{8}{3}) \\ &= 8 + 2B + \frac{16}{3} \quad \text{or} \quad B = -\frac{8}{3} \\ \therefore \quad \frac{8}{(x+1)(x^2+2)} &= \frac{8}{3(x+1)} + \frac{-\frac{8}{3}x + \frac{8}{3}}{x^2+2} \\ &= \frac{8}{3(x+1)} + \frac{-8x+8}{3(x^2+2)} \end{aligned}$$

(ii) The degree of the numerator is greater than the degree of the denominator, we resort to long division.
But since $(x+1)(x^4 - 2x) = x^5 + x^4 - 2x^2 - 2x$, then;

$$\begin{array}{r|llllll} & x & -1 & & & & \\ \hline x^5 + x^4 - 2x^2 - 2x & x^6 & & & & & -1 \\ & x^6 & +x^5 & & -2x^3 - 2x^2 & & \\ \hline & & -x^5 & & +2x^3 + 2x^2 & & \\ & & -x^5 & -x^4 & & +2x^2 + 2x & \\ \hline & & & +x^4 + 2x^3 & & & -2x - 1 \end{array}$$

$$\therefore \quad \frac{x^6 - 1}{(x+1)(x^4 - 2x)} = x - 1 + \frac{x^4 + 2x^3 - 2x - 1}{(x+1)(x^4 - 2x)}$$

Thus,

$$\frac{x^4 + 2x^3 - 2x - 1}{(x+1)(x^4 - 2x)} = \frac{x^4 + 2x^3 - 2x - 1}{x(x+1)(x^3 - 2)}$$

Now, set

$$\frac{x^4+2x^3-2x-1}{x(x+1)(x^3-2)} \equiv \frac{A}{x} + \frac{B}{x+1} + \frac{Cx^2+Dx+E}{x^3-2}$$

Multiplying both sides by $x(x+1)(x^3-2)$;

$$\begin{aligned} x^4+2x^3-2x-1 &\equiv A(x+1)(x^3-2)+Bx(x^3-2) \\ &\quad +(Cx^2+Dx+E)x(x+1) \\ x^4+2x^3-2x-1 &\equiv A(x+1)(x^3-2)+Bx(x^3-2) \\ &\quad +Cx^3(x+1)+Dx^2(x+1) \\ &\quad +Ex(x+1) \end{aligned}$$

If $x=-1$, we have;

$$\begin{aligned} (-1)^4+2(-1)^3-2(-1)-1 &= B(-1)(-3) \quad \text{i.e.} \quad 3B=0 \\ \therefore \qquad B &= 0 \end{aligned}$$

For $x=0$,

$$\begin{aligned} -1 &= A(-2) \\ -2A &= -1 \\ A &= \frac{1}{2} \end{aligned}$$

Now, put $x=1$ (say);

$$\begin{aligned} 1+2(1)^3-2-1 &= A(2)(-1)+B(-1)+C(2) \\ &\quad +D(2)+E(2) \\ 0 &= -2A-B+2C+2D+2E \\ &= -2(\frac{1}{2})-0+2C+2D+2E \\ \therefore \qquad 2C+2D+2E &= 1 \qquad\qquad (i) \end{aligned}$$

If $x = -2$ (say);

$$\begin{aligned}
(-2)^4 + 2(-2)^3 - 2(-2) - 1 &= A(-1)(-10) + B(-2)(-10) \\
&\quad +C(-8)(-1) + D(4)(-1) \\
&\quad +E(-2)(-1) \\
3 &= 10A + 20B + 8C - 4D + 2E \\
&= 10\left(\frac{1}{2}\right) + 0 + 8C - 4D + 2E \\
\therefore \quad 4C - 2D + E &= -1 \qquad (ii)
\end{aligned}$$

Put $x = 2$ (say);

$$\begin{aligned}
(2)^4 + 2(2)^3 - 2(2) - 1 &= A(3)(6) + B(2)(6) + C(8)(3) \\
&\quad +D(4)(3) + E(2)(3) \\
27 &= 18A + 12B + 24C + 12D + 6E \\
&= 18(\frac{1}{2}) + 0 + 24C + 12D + 6E \\
\therefore \quad 4C + 2D + E &= 3 \qquad (iii)
\end{aligned}$$

Solving equations (i), (ii) and (iii), we have; $C = \frac{1}{2}$, $D = 1$ and $E = -1$

$$\begin{aligned}
\therefore \quad \frac{x^4 + 2x^3 - 2x - 1}{(x+1)(x^4 - 2x)} &= \frac{1}{2x} + \frac{\frac{1}{2}x^2 + x - 1}{x^3 - 2} \\
&= \frac{1}{2x} + \frac{x^2 + 2x - 2}{2(x^3 - 2)}
\end{aligned}$$

Thus,

$$\frac{x^6 - 1}{(x+1)(x^4 - 2x)} = x - 1 + \frac{1}{2x} + \frac{x^2 + 2x - 2}{2(x^3 - 2)}$$

Case IV: Example 3.4.4

Express the following in partial fraction,

$$\frac{(4x^2 - 1)(x - 6)^2}{(x^2 - x + 1)^2(x + 1)}$$

Solution

The degree of the numerator is less than the degree of the denominator, therefore we set,

$$\frac{(4x^2-1)(x-6)^2}{(x^2-x+1)^2(x+1)} \equiv \frac{Ax+B}{(x^2-x+1)} + \frac{Cx+D}{(x^2-x+1)^2} + \frac{E}{x+1}$$

Multiplying through by $(x^2-x+1)^2(x+1)$;

$$\begin{aligned} (4x^2-1)(x-6)^2 \;\equiv\; & (Ax+B)(x^2-x+1)(x+1) \\ & +(Cx+D)(x+1) + E(x^2-x+1)^2 \end{aligned}$$

If $x = -1$,

$$\begin{aligned} 4[(-1)^2-1](-7)^2 &= E[(-1)^2-(-1)+1]^2 \\ &= 9E \\ \text{i.e.} \quad 147 &= 9E \quad \text{or} \quad E = \frac{49}{3} \\ \therefore \quad D &= -B - \frac{157}{3} \qquad\qquad (i) \end{aligned}$$

If $x = 0$ (say),

$$\begin{aligned} (-1)(-6)^2 &= B + D + E \\ B + D + E &= -36 \\ \text{i.e.} \quad B + D &= -36 - \frac{49}{3} \\ \therefore \quad B + D &= -\frac{157}{3} \qquad\qquad (ii) \\ \text{or} \quad D &= -B - \frac{157}{3} \end{aligned}$$

Putting $x = 1$,

$$\begin{aligned}
3(-5)^2 &= A(2) + B(2) + C(2) + D(2) + E \\
75 &= 2A + 2B + 2C + 2D + E \\
&= 2A + 2C + 2(B + D) + E \\
&= 2A + 2C + 2\left(-\frac{157}{3}\right) + \frac{49}{3} \\
2A + 2C &= \frac{490}{3} \\
A + C &= \frac{245}{3} \qquad\qquad (iii) \\
\text{or} \quad C &= -A + \frac{245}{3}
\end{aligned}$$

Put $x = -2$,

$$\begin{aligned}
15(-8)^2 &= A(-2)(7)(-1) + B(7)(-1) + C(-2)(-1) \\
&\quad + D(-1) + E(7)^2 \\
960 &= 14A - 7B + 2C - D + 49E \\
&= 14A - 7B + 2C - D + 49(\frac{49}{3}) \\
&= 14A - 7B + 2\left(\frac{245}{3} - A\right) - \left(\frac{-157 - 3B}{3}\right) + \frac{2401}{3} \\
&= 14A - 7B + 2\left(\frac{245 - 3A}{3}\right) - \left(\frac{-157 - 3B}{3}\right) + \frac{2401}{3}
\end{aligned}$$

$$\begin{aligned}
\therefore \qquad 2A - B &= -\frac{28}{3} \qquad\qquad (iv) \\
\text{or} \quad B &= 2A + \frac{28}{3}
\end{aligned}$$

Put $x = 2$,

$$\begin{aligned}
15(-4)^2 &= A(2)(3)(3) + B(3)(3) + C(2)(3) + D(3) + E(3)^2 \\
240 &= 18A + 9B + 6C + 3D + 9\left(\frac{49}{3}\right) \\
&= 18A + 9B + 6C + 3D + 147 \\
93 &= 18A + 9B + 6C + 3D \\
31 &= 6A + 3B + 2C + D \\
&= 6A + 3B + 2\left(\frac{245}{3} - A\right) + \left(-\frac{157}{3} - B\right) \\
-240 &= 12A + 6B \\
\text{i.e.} \quad 2A + B &= -40 \qquad\qquad (v)
\end{aligned}$$

Now, solving equations $(ii) - (v)$, we have;

$$A = -\frac{37}{3}, \quad B = -\frac{46}{3}, \quad C = 94, \quad D = -37$$

and since we have already got E to be $\frac{49}{3}$, we therefore have the resolved partial fraction as;

$$\frac{(4x^2-1)(x-6)^2}{(x^2-x+1)^2(x+1)} = \frac{-37x-46}{3(x^2-x+1)} + \frac{94x-37}{(x^2-x+1)^2} + \frac{49}{3(x+1)}$$

Chapter 4

Theory of Indices, Logarithms and Surds

This chapter is discussed here to lay the foundation and give a proper understanding to the concept and formulation of the Logarithm Tables, commonly found in Four Figure Tables and Electronic Calculators. Also, in performing basic calculations involving irrational numbers, overflows is usually encountered and the concept for the representation of the decimal numbers resulting from such calculations are given in standard notations using the fundamental concepts of the theory of indices. Furthermore, there are errors due to roundoff of numbers in decimal or standard notation resulting from irrational numbers; such errors can be avoided by performing your calculations in surd form. This is therefore, the basis for performing calculations using surd in elementary mathematics which we shall also discuss in this chapter.

4.1 Theory of Indices

In elementary arithmetic, we know that 5^3 means $5 \times 5 \times 5$, hence we write a^3 as $a \times a \times a$, and more generally, $a \times a \times a \times \ldots \times a$, (the product of a n-times) as a^n. Here a is referred to as the *base* and n is the *index*, *power* or *exponent*. The theory of indices involves the following rules (m and n being positive integers).

Rule I:

$$a^m \times a^n = a^{m+n} \tag{4.1.1}$$

The above rule in equation 4.1.1 can be extended as follows:

$$a^m \times a^n \times a^p \times \ldots = a^{m+n+p+\ldots}$$

where p is also a positive integer.

Example 4.1.1

$4^3 \times 4^5 = 4^{3+5} = 4^8$

Rule II:

$$a^m \div a^n = a^{m-n}, \qquad (m > n) \tag{4.1.2}$$

Example: 4.1.2

$6^5 \div 6^2 = 6^{5-2} = 6^3$

Rule III:

$$(a^m)^n = a^{mn} \text{ and } \left(\frac{a}{b}\right)^p = \frac{a^p}{b^p} \tag{4.1.3}$$

By extension of the above rule, we have;

$$\left(\frac{a^m b^n \times \ldots}{c^p \times \ldots}\right)^q = \frac{a^{mq} b^{nq} \times \ldots}{c^{pq} \times \ldots}$$

where $p, q \in \mathbb{Z}^+$

Example 4.1.3

$(7^3)^2 = 7^{3\times 2} = 7^6$

We have just discussed the situations where the index are positive integers; however, when the index are negative or irrational, such as $a^0, a^{-m}, a^{m/n}$, how do you treat such case(s)? We shall provide answers to these in the following section.

4.1.1 Negative and Fractional Indices

Theorem 4.1.1. *To find a meaning for a^0.*

Proof. By Rule II of indices above, $a^m/a^m = a^{m-m} = a^o$
But

$$\begin{aligned} a^m/a^m &= 1 \\ \therefore \quad a^0 &= 1 \end{aligned} \tag{4.1.4}$$

□

Example 4.1.4

Write down the value of 3^0.

Solution

$$3^0 = 3^m/3^m = 1$$

Theorem 4.1.2. *To find the meaning for a^{-m} where m is a positive number.*

Proof. By Rule I of indices,

$$\begin{aligned} a^m \times a^{-m} &= a^{m+(-m)} \\ &= a^{m-m} = a^0 \end{aligned}$$

But

$$\begin{aligned} a^0 &= 1 \\ \text{i.e.} \quad a^m \times a^{-m} &= 1 \\ \therefore \quad a^{-m} &= \tfrac{1}{a^m} \end{aligned} \tag{4.1.5}$$

□

Example 4.1.5

Write down the values of 3^{-3}, $\left(\frac{3}{8}\right)^{-1}$, $(-4)^{-3}$, $\left(\frac{2}{3}\right)^{-2}$

Solution

(i) $3^{-3} = \frac{1}{3^3} = \frac{1}{27}$.

(ii) $\left(\frac{3}{8}\right)^{-1} = \frac{1}{3/8} = \frac{8}{3}$

(iii) $(-4)^{-3} = \frac{1}{(-4)^3} = \frac{1}{-64} = -\frac{1}{64}$

(iv) $\left(\frac{2}{3}\right)^{-2} = \left(\frac{1}{2/3}\right)^2 = \left(\frac{3}{2}\right)^2 = \frac{9}{4}$

Theorem 4.1.3. *To find the meaning for $a^{m/n}$, where m and n are any integers (positive or negative).*

Proof. By Rule III of indices,

$$\begin{aligned} (a^{m/n})^n &= a^{m/n \times n} \\ &= a^m \end{aligned}$$

If we put $m = 1$;

$$\begin{aligned} (a^{1/n})^n &= a^{1/n \times n} \\ &= a^1 = a \\ \therefore \quad (a^{1/n})^n &= (\sqrt[n]{a})^n = a \end{aligned}$$

$$\begin{aligned} \therefore \quad a^{1/n} &= \sqrt[n]{a} \\ \text{Hence} \quad a^{m/n} &= (\sqrt[n]{a})^m = \sqrt[n]{a^m} \end{aligned} \tag{4.1.6}$$

□

Example 4.1.6

Calculate the values of the following:

$$8^{2/3},\ 16^{\frac{-3}{4}},\ \sqrt[3]{2^6},\ \left(\frac{144}{169}\right)^{-\frac{1}{2}}$$

Solution

(i)

$$\begin{aligned} 8^{2/3} &= (\sqrt[3]{8})^2 \\ &= 2^2 = 4 \end{aligned}$$

(ii)

$$\begin{aligned} 16^{-3/4} &= (\sqrt[4]{16})^{-3} \\ \frac{1}{16^{3/4}} &= \frac{1}{(\sqrt[4]{16})^3} \\ &= \frac{1}{2^3} \\ &= \frac{1}{8} \end{aligned}$$

(iii)

$$\begin{aligned} \sqrt[3]{2^6} &= (2^{1/3})^6 \\ &= 2^{1/3\times 6} \\ &= 2^2 = 4 \end{aligned}$$

(iv)

$$\begin{aligned} \left(\frac{144}{169}\right)^{-\frac{1}{2}} &= \left(\frac{1}{144/169}\right)^{\frac{1}{2}} \\ &= \left(\frac{169}{144}\right)^{\frac{1}{2}} \\ &= \sqrt{\frac{169}{144}} = \frac{13}{12} \end{aligned}$$

4.1.2 Standard Index Form (Scientific Notation)

Decimal numbers (particularly very large or very small numbers) are usually expressed in a uniform and more economical way based on the powers of 10. Such a number is said to be in *standard index form* or *scientific notation* and is of the form;

$$m \times r^e \ldots \qquad (4.1.7)$$

where m is the *mantissa*, such that $1 \leq m < 10$ and r is the *radix* or *base* of the arithmetic used and is usually equal to 10, while e is the *exponent* (positive or negative integer).

Example 4.1.7

Write the following in standard form, 0.000 003 4, 52 000 000.

Solution

(i) $0.000\ 003\ 4 = 3.4 \times 10^{-6}$

(ii) $52\ 000\ 000 = 5.2 \times 10^7$

Example 4.1.8

(i) Simplify the expression $\sqrt[3]{8x^{-3})^{-2}}$

Solution

$$\begin{aligned}
\sqrt[3]{(8x^{-3})^{-2}} &= (\sqrt[3]{2^3x^{-3}})^{-2} \\
&= \left(\sqrt[3]{\frac{2^3}{x^3}}\right)^{-2} \\
&= \left[\sqrt[3]{\left(\frac{2}{x}\right)^3}\right]^{-2} \\
&= \left(\frac{2}{x}\right)^{-2} \\
&= \left(\frac{x}{2}\right)^{2} \\
&= \frac{x^2}{4}
\end{aligned}$$

(ii) Simplify the expression $\sqrt{\frac{49x^4}{16}} \times \left(\frac{x}{2}\right)^{-3}$.

Solution

$$\begin{aligned}
\sqrt{\frac{49x^4}{16}} \times \left(\frac{x}{2}\right)^{-3} &= \sqrt{\left(\frac{7x^2}{4}\right)^2} \times \left(\frac{2}{x}\right)^3 \\
&= \frac{7x^2}{4} \times \frac{8}{x^3} \\
&= \frac{14}{x}
\end{aligned}$$

(iii) Simplify,

$$\frac{3^x \times 9^{x+1}}{27^{x-1}} \div 27$$

Solution

$$\begin{aligned}
\frac{3^x \times 9^{x+1}}{27^{x-1}} \div 27 &= \frac{3^x \times 3^{2(x+1)}}{27^{x-1}} \times \frac{1}{3^3} \\
&= \frac{3^{x+2x+2}}{3^{3x-3+3}} \\
&= \frac{3^{3x+2}}{3^{3x}} \\
&= 3^{3x+2-3x} \\
&= 3^2 = 9
\end{aligned}$$

(iv) Simplify $2^x \times 9^{x+1} \times 8^{x+1}$

Solution

$$\begin{aligned} 2^x \times 9^{x+1} \times 8^{x+1} &= 2^x \times 3^{2(x+1)} \times 2^{3(x+1)} \\ &= 2^{x+3x+3} \times 3^{2(x+1)} \\ &= 2^{4x+3} \times 3^{2x+2} \\ &= 2^{2(2x)+3} \times 3^{2x+2} \\ &= 4^{2x} \times 2^3 \times 3^{2x} \times 3^2 \\ &= (4 \times 3)^{2x} \times 8 \times 9 \\ &= 72 \times 12^{2x} \end{aligned}$$

(v) Simplify $2^x \times 4^{x-1} \times 8^{x+1}$

Solution

$$\begin{aligned} 2^x \times 4^{x-1} \times 8^{x+1} &= 2^x \times 2^{2(x-1)} \times 2^{3(x+1)} \\ &= 2^x \times 2^{2x-2} \times 2^{3x+3} \\ &= 2^{x+2x-2+3x+3} \\ &= 2^{6x+1} \end{aligned}$$

4.2 Logarithms

Definition 2 *The logarithm of* y *to the base* a *is written as*

$$\log_a y$$

In other words, the logarithm of a number to the base a is the power to which a must be raised to give the number. For example, if

$$\log_a y = x, \text{ then } a^x = y \tag{4.2.1}$$

It follows that, when a number is expressed as a power of a, then its logarithm to that base is equal to the index. In other words, the

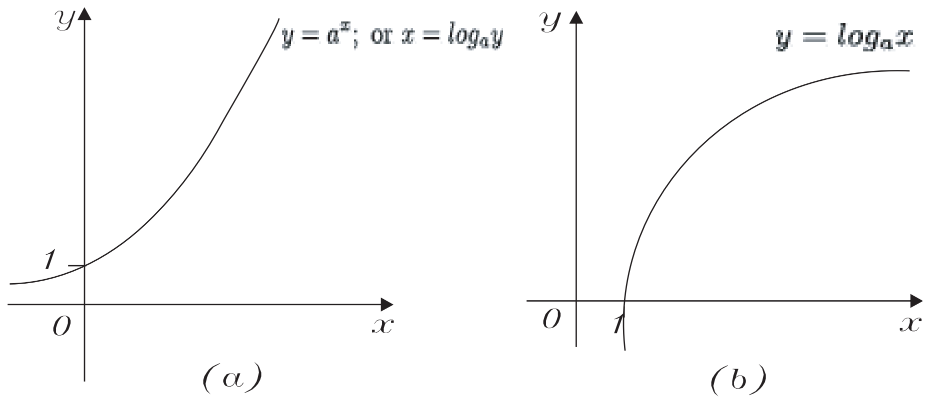

Figure 4.1: Curve of Logarithmic function

exponential function a^x is closely connected to the logarithmic function since one is the inverse of the other. The graphical illustration is as follows;

The Figure 4.1(a) and (b) indicates the close relation between the exponential and logarithmic functions.

Note from Figure 4.1 that:

(i) $\log_a 1 = 0$;

(ii) $\log_a x$ does not exist if x is negative;

(iii) The logarithm of a positive number less than 1 is always negative, i.e if x lies between 0 and 1, then $\log_a x$ is negative;

(iv) $\log_a 0$ is undefined;

(v) x is directly proportional to $\log_a x$.

Note that, when the base is 10, the logarithms are known as *common logarithms*, and conventionally, if the base is not given, it is understood to be 10. However, when the base is e $(e = 2.718\,2\ldots)$, the logarithm is known as *Natural or Napierian logarithm*

4.2.1 Operating Rules for Logarithms

Theorem 4.2.1. *To prove that* $\log_a(M \times N) = \log_a M + \log_a N$.

Proof. Let

$$M = a^x \qquad (i)$$

$\therefore$ by the definition of logarithm in (4.2.1),

$$\log_a M = x \qquad (ii)$$

Similarly, if

$$\begin{aligned} N &= a^y && (iii) \\ \therefore \quad \log_a N &= y && (iv) \end{aligned}$$

From $(i) \times (iii)$:

$$\begin{aligned} M \times N &= a^x \times a^y \\ &= a^{x+y} \quad \text{from} \quad (4.1.1) \end{aligned}$$

By taking the logarithm of both sides to base a;

$$\begin{aligned} \log_a(M \times N) &= \log_a a^{x+y} \\ &= (x+y)\log_a a \\ &= x + y. \end{aligned}$$

But from (ii) and (iv) it follows that;

$$\log_a(M \times N) = x + y = \log_a M + \log_a N \qquad (4.2.2)$$

□

Example 4.2.1

$$\log_a 10 = \log_a(5 \times 2) = \log_a 5 + \log_a 2$$

Theorem 4.2.2. *To prove that* $\log_a\left(\frac{M}{N}\right) = \log_a M - \log_a N$

Proof. Using the same notation as in the previous theorem.

$$\frac{M}{N} = \frac{a^x}{a^y} = a^{x-y} \qquad \text{(2nd rule of indices) - (4.1.2)}$$

By taking the logarithm of both sides to base a,

$$\log_a\left(\frac{M}{N}\right) = \log_a a^{x-y} = (x-y)\log_a a = x - y.$$

$$\therefore \qquad \log_a\left(\frac{M}{N}\right) = x - y = \log_a M - \log_a N \qquad (4.2.3)$$

□

Example 4.2.2

$$\begin{aligned} \log_a 7 &= \log_a\left(\frac{14}{2}\right) \\ &= \log_a 14 - \log_a 2 \end{aligned}$$

Theorem 4.2.3. *To prove that* $\log_a M^b = b\log_a M$.

Proof. Let

$$M = a^x \qquad (i)$$

By taking the logarithm of both sides to base a;

$$\begin{aligned} \log_a M &= \log_a a^x \\ &= x\log_a a = x \end{aligned}$$

From (i), $M^b = (a^x)^b = a^{xb}$ (3rd rule of indices - (4.1.3)).
Similarly, by taking the logarithm of both sides to base a, we obtain;

$$\log_a M^b = \log_a a^{xb} = bx = b\log_a M \qquad (4.2.4)$$

□

Examples 4.2.3

(i) $\log_2 16 = \log_2 2^4 = 4$

(ii) $\log_3 \frac{1}{27} = \log_3 3^{-3} = -3$

(iii) $\log_5 0.04 = \log_5 \frac{1}{25} = \log_5 5^{-2} = -2$

(iv) $\log_{\frac{1}{4}} 16 = \log_{\frac{1}{4}} \left(\frac{1}{4}\right)^{-2} = -2$

(v)

$$\begin{aligned}\log_7 7\sqrt{7} &= \log_7 7 \times 7^{\frac{1}{2}} \\ &= \log_7 7^{1+\frac{1}{2}} \\ &= \log_7 7^{\frac{3}{2}} \\ &= \frac{3}{2}\log_7 7 \\ &= \frac{3}{2}\end{aligned}$$

Theorem 4.2.4. *To prove that* $\log_p M = \log_a M/\log_a P$.

Proof. Let

$$\begin{aligned} M &= P^x \\ \therefore \quad \log_p M &= x \qquad (i)\end{aligned}$$

Taking the logarithms of both sides to base a in (i)

$$\begin{aligned}\log_a M &= \log_a P^x \\ &= x \log_a P \\ \therefore \quad x &= \log_a M/\log_a P\end{aligned}$$

$$\text{i.e.} \quad \log_P M = \log_a M/\log_a P \tag{4.2.5}$$

Now if $M = a$ in the result.

$$\log_P a = \frac{\log_a a}{\log_a P} = \frac{1}{\log_a P} \tag{4.2.6}$$

□

Note that the expression in equation (4.2.5) can be used for converting the logarithm to any base P to a common logarithm (quotient) by putting $a = 10$, in order to find the value and using the logarithm tables.

Example 4.2.4

If $17^x = 35$, find x.

Solution

$$
\begin{aligned}
17^x &= 35 \\
\text{i.e.} \quad \log 17^x &= \log 35 \\
x \log 17 &= \log 35 \\
\therefore \quad x &= \frac{\log 35}{\log 17} \\
&= \frac{1.544\ 068}{1.230\ 448\ 9} = 1.254\ 881\ 9
\end{aligned}
$$

Example 4.2.5

Evaluate $\log_{\sqrt{2}} 1.7$.

Solution

$$
\begin{aligned}
\log_{\sqrt{2}} 1.7 &= \frac{\log 1.7}{\log \sqrt{2}} \\
&= \frac{\log 1.7}{\log 2^{\frac{1}{2}}} \\
&= \frac{\log \frac{17}{10}}{\log 2^{\frac{1}{2}}} \\
&= \frac{\log 17 - \log 10}{\frac{1}{2} \log 2} \\
&= \frac{\log 17 - 1}{\frac{1}{2} \log 2} \\
&= \frac{1.230\ 448\ 9 - 1}{\frac{1}{2}(0.30103)} \\
&= \frac{0.230\ 448\ 9}{0.150\ 515} = 1.531\ 069\ 5
\end{aligned}
$$

Remark 7 *(i)* $\log_a 1 = 0$.

Observe that for any base $a, a^0 = 1$, *(rule I indices). It follows* *that,*

$$\begin{aligned} \log_a a^0 &= \log_a 1 \\ \therefore \quad 0 &= \log_a 1. \end{aligned} \tag{4.2.5}$$

(ii) $\log_a a = 1$
again

$$\begin{aligned} a^1 &= a \\ \log_a a^1 &= \log_a a \\ \therefore \quad 1 &= \log_a a \end{aligned} \tag{4.2.6}$$

4.3 Indicial and Logarithmic Equations

Example 4.3.1

If $4^{x+1} = 8^{x-1}$, find x.

Solution

$$\begin{aligned} 4^{x+1} &= 8^{x-1} \\ \text{i.e.} \quad 2^{2(x+1)} &= 2^{3(x-1)} \end{aligned}$$

Since they are of the same base,

$$\begin{aligned} 2(x+1) &= 3(x-1) \\ 2x+2 &= 3x-3 \\ 2x-3x &= -2-3 \\ \therefore \quad -x &= -5 \\ \text{or} \quad x &= 5 \end{aligned}$$

Example 4.3.2

Find x from the formula $e^x = 100$, given that $e = 2.718$.

Solution

$$\begin{aligned} e^x &= 100 \\ \log e^x &= \log 100 \\ x \log e &= 2 \log 10 \\ \therefore \qquad x &= \frac{2}{\log e} = \frac{2}{\log 2.718} = 4.605\ 7 \end{aligned}$$

Example 4.3.3

If $5^{x-1} = 4^{x-1}$, find x.

Solution

$$\begin{aligned} 5^{x-1} &= 4^{x-1} \\ 5^{x-1} &= 2^{2(x-1)} \\ \log 5^{x-1} &= \log 2^{2(x-1)} \\ (x-1)\log 5 &= (2x-2)\log 2 \\ x \log 5 - \log 5 &= 2x \log 2 - 2\log 2 \\ x(\log 5 - 2\log 2) &= \log 5 - 2\log 2 \\ \therefore \qquad x &= \frac{\log 5 - 2\log 2}{\log 5 - 2\log 2} = 1 \end{aligned}$$

Example 4.3.4

Solve the equation,

$$2 \times 3^{2x+3} - 7 \times 3^{x+1} - 68 = 0$$

Solution

Rewrite the equation;

$$\begin{aligned} 2(3^3 \times 3^{2x}) - 7(3 \times 3^x) - 68 &= 0 \\ 2 \times 3^3(3^{2x}) - 7 \times 3(3^x) - 68 &= 0 \\ 54(3^x)^2 - 21(3^x) - 68 &= 0 \end{aligned}$$

Now let $3^x = u$, then we have;

$$54u^2 - 21u - 68 = 0$$

which is a quadratic equation in u.
By completing the square;

$$\begin{aligned}
u^2 - \frac{21}{54}u &= \frac{68}{54} \\
\left(u - \frac{21}{108}\right)^2 &= \frac{34}{27} + \left(\frac{21}{108}\right)^2 \\
u - \frac{21}{108} &= \sqrt{\left(\frac{34}{27} + \frac{441}{11664}\right)} \\
\therefore \qquad u &= \left(\frac{34}{27} + \frac{441}{11664}\right)^{\frac{1}{2}} + \frac{21}{108} \\
&= \pm 1.138 + \frac{21}{108} \\
&= 1.333 \quad \text{or} \quad -0.944\,4
\end{aligned}$$

But $u = 3^x$,
Hence by using $u = 1.333$ or $\frac{4}{3}$, we have;

$$\begin{aligned}
\frac{4}{3} &= 3^x \\
\log\frac{4}{3} &= \log 3^x \\
\therefore \qquad x &= \frac{\log 4 - \log 3}{\log 3} \simeq 0.262
\end{aligned}$$

Since the logarithm of a negative number does not exist,

$$x \simeq 0.262$$

Example 4.3.5

Solve the equation,

(i) $\log_2(x^2 - 5x - 10) = 2$.

Solution

Rewrite the equation;

$$\begin{aligned}
x^2 - 5x - 10 &= 2^2 \\
x^2 - 5x - 14 &= 0 \\
(x-7)(x+2) &= 0 \\
\therefore \quad x &= 7 \quad \text{or} \quad -2
\end{aligned}$$

(ii) $\log_{10}(y^2 + 13y) = 1 + \log_{10}(1+y)$

Solution

$$\begin{aligned}
\log_{10}(y^2 + 13y) &= \log_{10} 10^1 + \log_{10}(1+y) \\
\log_{10}(y^2 + 13y) &= \log_{10}(10 \times (1+y)) \\
\log_{10}(y^2 + 13y) &= \log_{10}(10 + 10y) \\
y^2 + 13y &= 10 + 10y \\
y^2 + 3y - 10 &= 0 \\
(y+5)(y-2) &= 0 \\
\therefore \quad y &= -5 \quad \text{or} \quad 2
\end{aligned}$$

But since y cannot be negative, therefore $y = 2$

Example 4.3.6

Show that $a^{\log_a x} = x$, and hence evaluate $7^{-3} \log_7 2$.

Solution

Suppose,

$$\begin{aligned}
P &= a^{\log_a x} \\
\text{then,} \quad \log_a P &= \log_a a^{\log_a x} \\
\text{i.e.} \quad \log_a P &= \log_a x \\
\therefore \quad P &= x
\end{aligned}$$

But we already have,

$$\begin{aligned} P &= a^{\log_a x} \\ \therefore \quad x &= a^{\log_a x} \end{aligned}$$

Next, we know that,

$$\begin{aligned} 7^{-3\log_7 2} &= 7^{\log_7 2^{-3}} \\ \therefore \quad \text{if} \quad P &= 7^{\log_7 2^{-3}} \\ \text{then,} \quad \log_7 P &= \log_7 7^{\log_7 2^{-3}} \\ &= \log_7 2^{-3} \\ \therefore \quad P &= 2^{-3} = \frac{1}{8} \end{aligned}$$

Hence, we now have,

$$7^{-3\log_7 2} = \frac{1}{8}$$

which concludes the evaluation.

4.4 Theory of Surds

As stated in Section (1.4) a *rational number* is any number that can be expressed in the form a/b ($b \neq 0$), where a and b are integers, whilst any number, such as $\sqrt{3}$, $\sqrt[3]{5}$, $\sqrt[5]{7}$, etc., that is a real number which cannot be expressed in this form, is known as an *irrational* number.

Surd is usually used to denote an expression involving irrational numbers of the form $\sqrt[n]{b}$, where b is not a perfect n^{th} power of a rational number. Thus, square root of numbers or a combination of square roots and real number gives rise to surds. They are commonly used for a real number where square root cannot be found e.g. $\sqrt{5}, \sqrt{19}$ etc. Although, approximate values can be obtained from square root tables or electronic calculations, but it is often simpler and efficient to work with the surd themselves in order to reduce *roundoff error*.

In general, when dealing with *surd* quantities, only square roots will be encountered but it is possible for cube and higher roots to occur. However, we shall restrict ourselves in this section to *quadratic surds*, i.e. surds which are square roots only.

Lemma 4.4.1. *If R be any real number, then $R^{1/2} = \sqrt{R}$ (refer to Theorem III in section 4.1.1).*

Proof. Let

$$\sqrt{R} = r \qquad (i)$$

Then, $R = r^2$
Thus,

$$\begin{aligned} R^{1/2} &= (r^2)^{1/2} \\ &= r^{2.\frac{1}{2}} = r^1 = r \\ \text{i.e.,} \quad R^{\frac{1}{2}} &= r \qquad (ii) \end{aligned}$$

Comparing the results of (i) and (ii), we observe that;

$$R^{\frac{1}{2}} = \sqrt{R}$$

□

Lemma 4.4.2 (nth Root Theorem) . *If R is a given real number, then $R^{1/n} = \sqrt[n]{R}$, where n is any given real number. (Refer to Theorem III in section 4.1.1.)*

Proof. Let

$$\sqrt[n]{R} = r \qquad (i)$$

Then, $R = r^n$.
It follows that,

$$\begin{aligned} R^{1/n} &= (r^n)^{1/n} \\ &= r^{n \times \frac{1}{n}} = r^1 = r \\ \text{i.e.} \quad R^{\frac{1}{n}} &= r \end{aligned}$$

Hence, by the relations of (i) and (ii), we conclude that,

$$R^{\frac{1}{n}} = \sqrt[n]{R}$$

□

4.4.1 Arithmetic of Surds

(i) Reduction To Basic Form

Let a be a real number, then $\sqrt{a}$ is in basic form if a does not contain a factor which is a perfect square, i.e. $\sqrt{7}, \sqrt{37}$ etc., are in basic form and cannot be simplified any further.

Example 4.4.1

Simplify (i) $\sqrt{125}$, (ii) $\sqrt{8}$

Solution

(i) $\sqrt{125} = \sqrt{(25 \times 5)} = 5\sqrt{5}$

(ii) $\sqrt{8} = \sqrt{(4 \times 2)} = 2\sqrt{2}$

Rules for Surds

Let a and b be any real number then;
Rule I:

$$\sqrt{ab} = \sqrt{a} \times \sqrt{b} \tag{4.4.1}$$

Rule II:

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}, \quad b \neq 0 \tag{4.4.2}$$

Example 4.4.2

(i) $\sqrt{4 \times 36} = \sqrt{4} \times \sqrt{36} = 2 \times 6 = 12.$

(ii) $\sqrt{\frac{169}{49}} = \frac{\sqrt{169}}{\sqrt{49}} = \frac{13}{7}$

(ii) Addition and Subtraction of Similar Surds

Note that the Rules (4.4.1.) & (4.4.2) of surds do not apply as in the following case: $\sqrt{(a \pm b)} \neq \sqrt{a} \pm \sqrt{b}$, a and b are real number. Consequently addition and subtraction can only take place among surds in the same basic form.

Example 4.4.3

Simplify

(i) $\sqrt{28}+\sqrt{63}$,

(ii) $(5+\sqrt{6})-(3-2\sqrt{6})$

(iii) $(3-\sqrt{27})+(\sqrt{81}-5)$

Solution

(i)

$$\begin{aligned}\sqrt{28}+\sqrt{63} &= \sqrt{(4\times 7)}+\sqrt{(9\times 7)}\\ &= (\sqrt{4}\times\sqrt{7})+(\sqrt{9}\times\sqrt{7})\\ &= 2\sqrt{7}+3\sqrt{7}\\ &= (2+3)\sqrt{7}=5\sqrt{7}\end{aligned}$$

(ii)

$$\begin{aligned}(5+\sqrt{6})-(3-2\sqrt{6}) &= 5+\sqrt{6}-3+2\sqrt{6}\\ &= 5-3+\sqrt{6}+2\sqrt{6}\\ &= 2+(1+2)\sqrt{6}=2+3\sqrt{6}\end{aligned}$$

(iii)

$$\begin{aligned}(3-\sqrt{27})+(\sqrt{81}-5) &= 3-\sqrt{(9\times 3)}+\sqrt{(9\times 9)}-5\\ &= 3-3\sqrt{3}+9-5\\ &= 7-3\sqrt{3}\end{aligned}$$

4.4.2 Rationalization of Surds

Surds in the form such as $\frac{a}{\sqrt{b}}$, where a and b are real numbers and $b\neq 0$, can be written in more simplified form. This is done by multiplying both the numerator and denominator by the irrational number $(\sqrt{b})$ of the denominator. This process converts the irrational number in the denominator to rational number;

$$\frac{a\times\sqrt{b}}{\sqrt{b}\times\sqrt{b}}=\frac{a\sqrt{b}}{b} \tag{4.4.3}$$

This method is known as *rationalizing the denominator*

Example 4.4.4

Simplify (i) $\frac{6}{\sqrt{3}}$, (ii) $\frac{2}{\sqrt{5}}$, (iii) $\frac{3\sqrt{2}}{\sqrt{3}}$ by rationalizing the denominator.

Solution

(i) $\frac{6}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{6\sqrt{3}}{3} = 2\sqrt{3}$

(ii) $\frac{2}{\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}} = \frac{2\sqrt{5}}{5}$

(iii) $\frac{3\sqrt{2}}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{3\sqrt{6}}{3} = \sqrt{6}$

4.4.3 Conjugate Surds

If $a + \sqrt{b}$ or $\sqrt{a} + \sqrt{b}$ represents surd quantities, then $a - \sqrt{b}$ and $\sqrt{a} - \sqrt{b}$ are known as the *conjugate surds* respectively, and the products of the surd and their conjugate are;

$$(a + \sqrt{b}) \times (a - \sqrt{b}) = a^2 - b \qquad (4.4.4)$$

and,

$$(\sqrt{a} + \sqrt{b}) \times (\sqrt{a} - \sqrt{b}) = a - b \qquad (4.4.5)$$

which are rational numbers provided a and b are rational. So, when simplifying a fraction involving a surdic denominator, it is necessary to multiply numerator and denominator by the conjugate surd of the denominator, thus obtaining a rational denominator.

Example 4.4.5

Simplify,

(i) $\frac{3}{3-\sqrt{3}}$

(ii) $\frac{1}{\sqrt{2}-3}$

(iii) $\frac{4}{2\sqrt{2}+\sqrt{7}}$

(iv) $\frac{\sqrt{7}-3}{\sqrt{7}+3} + \frac{\sqrt{7}+3}{\sqrt{7}-3}$

(v) $\frac{1}{(\sqrt{7}-\sqrt{6})^2}$

Solution

(i) The conjugate of $3-\sqrt{3}$ is $3+\sqrt{3}$. So,

$$\begin{aligned}\frac{3}{3-\sqrt{3}} \times \frac{3+\sqrt{3}}{3+\sqrt{3}} &= \frac{3(3+\sqrt{3})}{9-3} \\ &= \frac{3(3+\sqrt{3})}{6} \\ &= (3+\sqrt{3})/2\end{aligned}$$

(ii) The conjugate of $\sqrt{2}-3$ is $\sqrt{2}+3$. So,

$$\begin{aligned}\frac{1}{\sqrt{2}-3} \times \frac{\sqrt{2}+3}{\sqrt{2}+3} &= \frac{\sqrt{2}+3}{2-9} \\ &= \frac{\sqrt{2}+3}{-7} \\ &= -\frac{3+\sqrt{2}}{7}\end{aligned}$$

(iii) The conjugate of $2\sqrt{2}+\sqrt{7}$ is $2\sqrt{2}-\sqrt{7}$. So,

$$\begin{aligned}\frac{4}{2\sqrt{2}+\sqrt{7}} \times \frac{2\sqrt{2}-\sqrt{7}}{2\sqrt{2}-\sqrt{7}} &= \frac{4(2\sqrt{2}-\sqrt{7})}{8-7} \\ &= 4(2\sqrt{2}-\sqrt{7})\end{aligned}$$

(iv) The conjugate of $\sqrt{7}+3$ is $\sqrt{7}-3$. So,

$$\begin{aligned}\frac{\sqrt{7}-3}{\sqrt{7}+3} + \frac{\sqrt{7}+3}{\sqrt{7}-3} &= \frac{\sqrt{7}-3}{(\sqrt{7}+3)} \times \frac{\sqrt{7}-3}{\sqrt{7}-3} + \frac{\sqrt{7}+3}{\sqrt{7}-3} \times \frac{\sqrt{7}+3}{\sqrt{7}+3} \\ &= \frac{(\sqrt{7}-3)^2}{7-9} + \frac{(\sqrt{7}+3)^2}{7-9} \\ &= [(7-6\sqrt{7}+9)+(7+6\sqrt{7}+9)]/(-2) \\ &= (16+16)/(-2) \\ &= -16\end{aligned}$$

(v) Expand the denominator and rationalize as follows:

$$
\begin{aligned}
\frac{1}{(\sqrt{7}-\sqrt{6})^2} &= \frac{1}{7-\sqrt{7\times 6}-\sqrt{7\times 6}+6} \\
&= \frac{1}{13-2\sqrt{42}} \\
&= \frac{1}{13-2\sqrt{42}} \times \frac{13+2\sqrt{42}}{13+2\sqrt{42}} \\
&= \frac{13+2\sqrt{42}}{169-168} \\
&= \frac{13+2\sqrt{42}}{1} \\
&= 13+2\sqrt{42}
\end{aligned}
$$

4.4.4 Equality of Surds

Theorem 4.4.1. *If two surdic quantities* $(a+\sqrt{b})$ *and* $(c+\sqrt{d})$ *are equal, then* $a=c$ *and* $b=d$*, where* a, b, c, d *are rational and* a *and* c *are not perfect squares.*

Proof. Let

$$
\begin{aligned}
a+\sqrt{b} &= c+\sqrt{d} \\
\text{then,} \quad a-c &= \sqrt{d}-\sqrt{b}
\end{aligned}
$$

But $a-c$ is not a surd, while $\sqrt{d}-\sqrt{b}$ is; i.e., a rational number = an irrational number, which is impossible unless both sides are equal to zero.

$$
\begin{aligned}
\therefore \quad a-c &= 0 \quad \text{and } \sqrt{d}-\sqrt{b} = 0 \\
\text{i.e.} \quad a &= c \quad \text{and} \quad \sqrt{d} = \sqrt{b} \\
\therefore \quad a &= c \quad \text{and} \quad d = b
\end{aligned}
$$

We already know that the product of zero and any finite number is zero. Thus it is clear from the nature of rational and irrational numbers, that a rational number can only be equal to an irrational number if each is zero, which is a neutral number (i.e. can be considered as being either rational or irrational). □

Theorem 4.4.2. *To find the square roots of the surdic quantities $a \pm \sqrt{b}$, where a and b are rational and b is not a perfect square.*

Proof. Let

$$\begin{aligned} \sqrt{(a \pm \sqrt{b})} &= \sqrt{x} \pm \sqrt{y}, \ (x, y \text{ rational }) \\ \text{Squaring,} \quad a \pm \sqrt{b} &= (\sqrt{x} \pm \sqrt{y})^2 \\ &= x \pm 2\sqrt{xy} + y \\ &= x + y \pm 2\sqrt{xy} \end{aligned}$$

From the previous theorem,

$$\begin{aligned} a &= x + y \qquad (i) \\ \sqrt{b} &= 2\sqrt{xy} \\ &= \sqrt{4xy} \\ \therefore \quad b &= 4xy \qquad (ii) \end{aligned}$$

Thus, we can use this to find the square root of a surd expression or quantity. The equations (i) and (ii) can be solved for x and y, by inspection or by algebraic means. □

Example 4.4.6

Find the positive square roots of; (i) $3 + 2\sqrt{2}$
(ii) $35 - 12\sqrt{6}$

(i) Let

$$\begin{aligned} \sqrt{(3 + 2\sqrt{2})} &= \sqrt{x} + \sqrt{y} \\ \text{Squaring,} \quad 3 + 2\sqrt{2} &= (\sqrt{x} + \sqrt{y})^2 \\ &= x + 2\sqrt{x}y + y \\ &= x + y + 2\sqrt{xy} \end{aligned}$$

Hence,

$$\begin{aligned} 3 &= x + y && (i) \\ \text{and} \quad 2\sqrt{2} &= 2\sqrt{xy} \\ \sqrt{8} &= \sqrt{4xy} \\ \therefore \quad 8 &= 4xy \\ \text{or} \quad 2 &= xy && (ii) \end{aligned}$$

But from (i), $y = 3 - x$
Using this in equation (ii)

$$\begin{aligned} 2 &= x(3 - x) = 3x - x^2 \\ \text{or} \quad x^2 - 3x + 2 &= 0 \\ (x - 2)(x - 1) &= 0 \\ \therefore \quad x &= 2 \text{ or } 1 \end{aligned}$$

Now, from equation (i), using these values;

$$\begin{aligned} y &= 3 - 2 \quad \text{or} \quad 3 - 1 \\ &= 1 \quad \text{or} \quad 2. \end{aligned}$$

Since the root must be positive, then $x > y$, and

$$\therefore \quad x = 2, \text{ and } y = 1$$

Hence, $\sqrt{(3 + 2\sqrt{2})} = \sqrt{2} + \sqrt{1} = \sqrt{2} + 1 = 1 + \sqrt{2}$.

(ii) Let

$$\sqrt{(35 - 12\sqrt{6})} = \sqrt{x} - \sqrt{y}; \ x > y$$

Squaring,

$$\begin{aligned} 35 - 12\sqrt{6} &= (\sqrt{x} + \sqrt{y} \\ &= x + y - 2\sqrt{xy} \\ \text{Hence} \quad 35 &= x + y && (i) \\ \text{and} \quad 12\sqrt{6} &= 2\sqrt{xy} \\ \sqrt{864} &= \sqrt{4}xy \\ 864 &= 4xy \\ \therefore \quad 216 &= xy && (ii) \end{aligned}$$

But from (i),

$$y = 35 - x \qquad (iii)$$

By substituting for y of (iii) in (ii),

$$\begin{aligned} 216 &= x(35 - x) \\ &= 35x - x^2 \\ \therefore \quad x^2 - 35x + 216 &= 0 \end{aligned}$$

Thus, by completing the square;

$$\begin{aligned} \left(x - \frac{35}{2}\right)^2 &= -216 + \left(\frac{35}{2}\right)^2 \\ &= -216 + \frac{1225}{4} \\ x - \frac{35}{2} &= \left(\frac{361}{4}\right)^{\frac{1}{2}} \\ &= \pm\frac{19}{2} \\ \therefore \quad x &= \pm\frac{19}{2} + \frac{35}{2} \\ &= 27 \quad \text{or} \quad 8 \end{aligned}$$

By substituting these values in equation (iii), we obtain;

$$\begin{aligned} y &= 35 - 27 \quad \text{or} \quad 35 - 8 \\ &= 8 \quad \text{or} \quad 27 \end{aligned}$$

But $x > y$, so $x = 27$ and $y = 8$.
Hence,

$$\begin{aligned} \sqrt{(35 - 12\sqrt{6})} &= \sqrt{27} - \sqrt{8} \\ &= 3\sqrt{3} - 2\sqrt{2} \end{aligned}$$

Chapter 5

Basic Set Theory and Relations

Set theory is the foundation stone of the edifice of modern mathematics. The precise definitions of all mathematical concepts are based on set theory. If one is to make any progress in higher mathematics itself or in its practical applications, one has to become familiar with the basic concepts and results of set theory and with the language in which they are expressed.

The knowledge of Set Theory helps us to classify items/elements into groups, subgroups and classes according to certain common attributes. For instance, in a given university the population of the school can be classified into sets of boys, girls, men and women. Similarly, in our everyday life, we are engaged in one form of classification or the other in our homes, offices or business activities. As earlier mentioned, classification can only be possible if there is one form of relationship or the other between the elements in a group, such attributes is the basis for the concept of Relations which we shall discuss later in this chapter – a father is related to the son because they are members of one family.

Indeed, set theory proved a unifying force giving all branches of mathematics a common basis, and their concepts a new clarity and precision. In this chapter, we shall emphasize those parts of set theory that have particularly important applications in the development of the various branches of mathematics.

There are basically two approaches to set theory; namely the *intuitive approach* and the *axiomatic approach*. However, we shall restrict ourselves to the intuitive approach in our work here, while interested readers may consult a more advanced text for further discussion on the later's approach.

5.1 Sets

We intuitively conceive of a *set* S as any well-defined collection of objects which we call *elements* of S. By well-defined, we mean that we must be able to decide definitely whether any one object does or does not belong to that set.

By this approach, the notion of a set S can be regarded as being synonymous with that of a *collection*, *class*, or *family* of objects called *elements* of S. Generally, we will use upper case letter, $A, B, C, \ldots$ to denote sets, and lower case letters, $a, b, c, \ldots$ to denote the elements of the sets. The statement 'a is an element of S' or equivalently, 'a belongs to S', is written $a \in S$ and if a is not in S (i.e. the negation of $a \in S$) is written $a \notin S$

Essentially, there are two ways of specifying a particular set. One way if it is possible, is to list its members. For example, V the set of all vowels in English Alphabet can be written as

$$V = \{a, e, i, o, u\}$$

Observe that the elements are separated by commas and enclosed in brace brackets. The second way is to state the rule or property which characterizes the elements in the set. Thus V can formally be described as the set of x such that x is a letter in the English alphabet and x is a vowel; which in shorthand notation is written as

$$V = \{x : x \text{ is a letter in the English alphabet, is a vowel}\}$$

or,

$$V = \{x | x \text{ is a letter in the English alphabet, } x \text{ is a vowel}\}$$

Note that a letter, usually x, is used to denote a typical member of the set; the colon or slash is read as 'such that' and the comma as 'and'. We also remark that in listing elements of a set, the order of the elements is immaterial. Thus

$$\{a, e, i, o, u\} \equiv \{e, u, a, i, o\}$$

In other words two sets are *equal* if and only if they have the same members irrespective of the ordering of the element in the set, and we write $A \neq B$ as the sets A and B are not equal. We shall give some examples of sets S to bring home our point.

(i) The set $\mathbb{R}$ of all real numbers,

$$\mathbb{R} = \{-3, -2, -1, 0, 1, 2, 3, \ldots\}$$

(ii) The set of integers, $\mathbb{Z}$, which are solutions of the quadratic equation,

$$\mathbb{Z} = \{x : x^2 - 2x - 3 = 0\},$$

sometimes called the *solution set* of the equation are 3 and -1, we could write, $\mathbb{Z} = \{3, -1\}$

(iii) The set of the ages of students in a class.

5.1.1 Finite and Infinite Sets

Consider a set $A = \{x : x \text{ is an integer}, 0 < x < 10\}$. Clearly, we could tell at once whether an integer is a member of the set A or not and we could list all the elements. In other words, a set A is said to be *finite* if the process of counting elements of A terminates. In particular, the set consisting of a single object (element) is called a *singleton* set e.g.

$$A = \{x : 3x - 2 = 0\}$$

is a singleton set. Otherwise, we say that the set A is *infinite*, if it is not practical to list all the elements of the set. For example, the set of integers

$$\mathbb{Z} = \{x : x \text{ is an integer}, x > 0\}$$

5.1.2 Universal Set, Empty Set

In the application of set theory, the members of all sets under investigation usually belong to some fixed large set called *universal set* or *universe of discourse* and is denoted by the symbol $\mathcal{U}$ unless otherwise stated or implied. The universal set is not unique as it changes from problem to problem, so that it must be defined for each particular problem of discourse. For example, in students population studies in a school, the universal set $\mathcal{U}$, consists of all the students in the school. On the other hand, a set is said to be *empty* or *null* if it contains no element and is usually denoted by the symbol $\emptyset$ or $\{\}$.
For example the set,

$$S = \{x : x \text{ is a real number}, x^2 + 4 = 0\} = \emptyset$$

has no elements, since $x^2 + 4 = 0$ has no real roots.

5.1.3 Subsets

Let A and B be two arbitrary sets. A set B is called a subset of a set A if every element of B is also an element of A. We also say that B is *contained in* A or that A *contains* B. Symbolically, we represent the relationship respectively as

$$B \subseteq A, \text{ or } A \supseteq B$$

The symbol $\subseteq$ stands for 'is a subset of or is equal to'. In other words, this definition implies that every set is a subset of itself. In particular, if $B \subseteq A$ and $A \subseteq B$, then $A = B$. i.e., two sets A, B are *equal* if and only if the above condition holds. However, if B is a subset of A and there exists at least one element of A that is not in B, (i.e. $B \neq A$) then we say that B is a *proper* subset of A and symbolically, we write.

$$B \subset A \text{ or } A \supset B$$

Conversely, if B is not a subset of A, i.e. if at least one element of B does not belong to A, we write,

$$B \not\subset A \quad \text{or} \quad A \not\supset B$$

Example 5.1.1

(i) The set of positive integers is a proper subset of the set of real numbers.

(ii) Consider the sets, $A = \{1, 4, 6, 8\}$, $\quad B = \{1, 2, 3, 4, 5, 6, 7, 8\}$ Then $A \subset B$ i.e. A is a proper subset of B or $A \supset B$.

Remark 8 *Every set A is a subset of the universal set $\mathcal{U}$ since, by definition all the members of A belong to $\mathcal{U}$. Also, the empty or null set $\emptyset$ is a subset of any set say A. If every element of a set A belongs to a set B, and every element of B belongs to a set C, then clearly, every element of A belongs to C. We state the above results in the following corollary.*

Corollary 5.1.1.

(i) For any arbitrary set A, we have $\emptyset \subset A \subset \mathcal{U}$.

(ii) For any arbitrary set A, we have $A \subset A$.

(iii) If $A \subset B$ and $B \subset C$, then $A \subset C$.

(iv) $A = B$ if and only if $A \subset B$ and $B \subset A$.

5.1.4 Complement of a Set

Let $\mathcal{U}$ be the universal set, and let A be a subset of $\mathcal{U}$, then the *compliment* of A is the set of all elements in $\mathcal{U}$ that are not in A denoted by A' or A^c and written as,

$$A' = \{x : x \in \mathcal{U},\ x \notin A\}$$

The *relative compliment* of a set B with respect to a set A or, simply, the *difference* of A and B denoted by $A - B$ or A/B, is the set of elements which belong to A but which does not belong to B:

$$A - B = \{x : x \in A, x \notin B\}$$

read A minus B

Example 5.1.2

$$\begin{aligned} \text{Let } \mathcal{U} &= \{1,2,3,4\},\ A = \{1,3\} \\ \text{then } A' &= \{2,4\} \quad \text{i.e.} \quad \mathcal{U} - A \end{aligned}$$

5.1.5 Union of Sets

Let A and B be two sets. We define the *union* of A and B, written as $A \cup B$, as the set of elements which are either in A or B, or in both. Thus,

$$A \cup B = \{x : x \in A \ \text{ or } \ x \in B\}$$

It follows from this definition that,

$$A \cup B = B \cup A$$

Example 5.1.3

Let,

$$A = \{1,2,3,4,5\},\ B = \{1,3,5,7,9\}$$

Then,

$$A \cup B = \{1,2,3,4,5,7,9\}$$

In particular, for any set A, $A \cup A' = \mathcal{U}$

5.1.6 Intersection of Sets

Suppose we have two arbitrary sets A and B. The *intersection* of A and B is the set of all elements which belong to both A and B and is denoted by $A \cap B$. Thus,

$$A \cap B = \{x : x \in A, x \in B\}$$

We observe that if $B \subset A$, then $A \cap B = B$. Also if the intersection of A and B is empty, then the two sets are said to be *disjoint* i.e.,

$$\emptyset = \{x : x \notin A, x \notin B\} = A \cap B$$

In particular for any arbitrary set A,

$$A \cap A' = \emptyset$$

Example 5.1.4

(i) Let $A\{1,2,3,4,5\}$, $B=\{1,2,3,5,7,9\}$
Then,

$$A \cap B = \{1,2,3,5\}$$

(ii) Let $A=\{1,2,3,4\}$, $B=\{x \in \mathbb{Z} : x^3 - 4x^2 - x + 4 = 0\}$
Then,

$$A \cap B = \{1,4\}$$

5.2 Venn Diagrams

Many verbal statements can be translated into equivalent statements about sets which can be described by Venn diagrams. A *Venn diagram* is a pictorial representation of sets by sets of points in the plane. The universal set $\mathcal{U}$ is represented by the interior of a rectangle, and the other sets are represented by circles lying within the rectangle. Thus, if A and B are two sets;

(i) If $B \subset A$, implies that the circle representing B will be entirely within the circle representing A as in Figure 5.1.

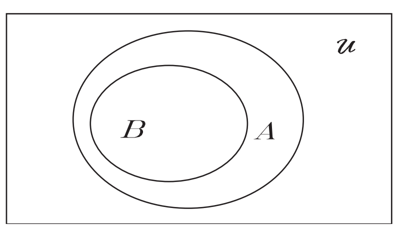

Figure 5.1: $B \subset A$ or $A \supset B$

(ii) If A and B are disjoint i.e. have no elements in common, then the circle representing A will be separated from the circle representing B.

(iii) Assuming in the two sets, A and B, we have the following possibilities:

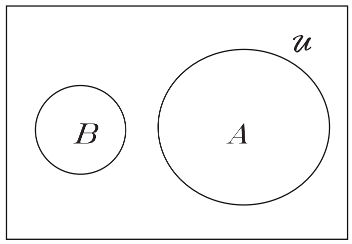

Figure 5.2: Empty set represented by $A \cap B = \emptyset$

(a) Some objects are in A but not in B,

(b) Some objects are in B but not in A,

(c) Some objects are in both A and B,

(d) Some objects are in neither A nor B; then we can represent A and B generally as in Figure 5.3.

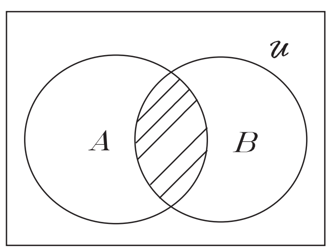

Figure 5.3: The shaded portion represents $A \cap B$

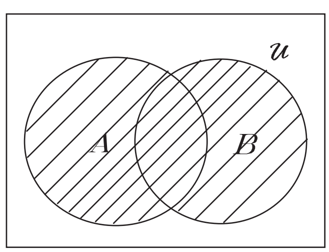

Figure 5.4: The shaded portion represents $A \cup B$

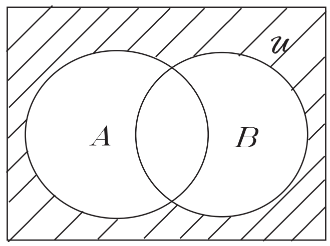

Figure 5.5: The shaded portion represents $A' - B \equiv B' - A$ or $(A \cup B)' \equiv A' \cap B'$

5.3 Laws of the Algebra of Sets

Let $\mathcal{U}$ be the universal set and let A, B, and C be arbitrary sets, under the above operations of union, intersection, and complement, such that the following holds;

(i) Idempotent Laws:

$$\begin{aligned} (a) \quad A \cup A &= A \\ (b) \quad A \cap A &= A \end{aligned}$$

(ii) Associative Laws:

$$\begin{aligned} (a) \quad (A \cup B) \cup C &= A \cup (B \cup C) \\ (b) \quad (A \cap B) \cap C &= A \cap (B \cap C) \end{aligned}$$

(iii) Commutative Laws:

$$\begin{aligned} (a) \quad A \cup B &= B \cup A \\ (b) \quad A \cap B &= B \cap A \end{aligned}$$

(iv) Distributive Laws:

$$\begin{aligned} (a) \quad A \cup (B \cap C) &= (A \cup B) \cap (A \cup C) \\ (b) \quad A \cap (B \cup C) &= (A \cap B) \cup (A \cap C) \end{aligned}$$

(v) Identity Laws:

$$\begin{array}{llcllcl} (a) & A \cap \mathcal{U} & = & A; & A \cup \emptyset & = & A \\ (b) & A \cap \emptyset & = & \emptyset; & A \cup \mathcal{U} & = & \mathcal{U} \end{array}$$

(vi) Involution Law: $(A')' = A$

(vii) Complement Laws:

$$\begin{array}{llcllcl} (a) & A \cup A' & = & \mathcal{U}; & A \cap A' & = & \emptyset \\ (b) & \mathcal{U}' & = & \emptyset', & \emptyset' & = & \mathcal{U} \end{array}$$

(viii) DeMorgan's Laws:

$$\begin{array}{llcl} (a) & (A \cup B)' & = & A' \cap B' \\ (b) & (A \cap B)' & = & A' \cup B' \end{array}$$

(These laws are synonymous to those of the algebra of propositions that we shall discuss in chapter 13. The similarity is rooted in the analogy between the set operations $\cup, \cap$, and complement, and the logical connective $\vee, \wedge$, and $\sim$).

For clarity, we shall give the proof of the Distributive and DeMorgan's Laws while the proofs of the remaining laws will be left as exercises.

(i) Proof of the Distributive Laws:

$$\begin{array}{rcl} A \cap (B \cup C) & = & (A \cap B) \cup (A \cap C) \\ L.H.S. & = & \{x : x \in A \text{ and } x \in (B \cup C)\} \\ & = & \{x : x \in A \text{ and } x \in B \text{ or } x \in C\} \\ & = & \{x : x \in A \text{ and } x \in B \text{ or } x \in A \text{ and } x \in C\} \\ & = & \{x : x \in (A \cap B) \text{ or } x \in (A \cap C)\} \\ & = & (A \cap B) \cup (A \cap C) \\ R.H.S. & = & (A \cap B) \cup (A \cap C) \\ & = & \{x : x \in (A \cap B) \text{ or } x \in (A \cap C)\} \\ & = & \{x : x \in A \text{ and } x \in B \text{ or } x \in A \text{ and } x \in C\} \\ & = & \{x : x \in A \text{ and } x \in B \text{ or } x \in C\} \\ & = & \{x : x \in A \text{ and } x \in (B \cup C)\} \\ & = & (A \cap (B \cup C) \end{array}$$

$$\therefore \quad A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \qquad (5.3.1$$

(ii) Proof of DeMorgan's Laws: $(A \cup B)' = A' \cap B'$
We need to show that,

$$(A \cup B)' \subset A' \cap B' \text{ and } A' \cap B' \subset (A \cup B)'$$

First, Let $x \in (A \cup B)'$, then $x \notin (A \cup B)$
That is x belongs to neither A nor B
i.e., $x \notin A$ or $x \notin B$ i.e., $x \in A'$ and $x \in B'$
Thus,

$$x \in A' \cap B'$$

Therefore, $(A \cup B)' \subset A' \cap B'$

Similarly, let $x \in A' \cap B'$, then, $x \in A'$ and $x \in B'$

$$\text{i.e.} \quad x \notin A \text{ or } x \notin B$$

Hence,

$$x \notin (A \cup B)$$

Since x is neither in A nor in B, then,

$$x \in (A \cup B)'$$

Therefore,

$$A' \cap B' \subset (A \cup B)'$$

Thus,

$$(A \cup B)' \subset A' \cap B' \text{ and } A' \cap B' \subset (A \cup B)'$$

$$\therefore \quad (A \cup B)' = A' \cap B' \tag{5.3.2}$$

The proof of $(A \cap B)' = A' \cup B'$ follows from above.

Theorem 5.3.1. *Let A and B be two arbitrary sets, then,*

(i) $(A - B) \subset A$

(ii) $(A - B) \cap B = \emptyset$

Proof.

(i) Let $x \in (A - B)$
By definition, $x \in A$ and $x \notin B$.
Now, since $x \in A$, it follows that;

$$(A - B) \subset A \tag{5.3.3}$$

(ii) Let $x \in (A - B) \cap B$
Thus $x \in A - B$ and $x \in B$
But $x \in A - B$, implies that

$$x \in A \text{ and } x \notin B$$

which contradicts our earlier assumption that $x \in B$. Hence, there does not exist any element in $(A - B) \cap B$
Therefore,

$$(A - B) \cap B = \emptyset \tag{5.3.4}$$

□

Theorem 5.3.2. *Let A and B be two disjoint finite sets and let $n(A)$ and $n(B)$ denote the number of elements of A and B respectively. Then $A \cup B$ is finite and*

$$n(A \cup B) = n(A) + n(B).$$

Proof. In counting the elements of $A \cup B$, first count those that are in A. There are $n(A)$ of elements in the finite set A. Next, the only other elements of $A \cup B$ are those that are in B but not in A. Nevertheless, since A and B are disjoint, no element of B is a member of A, so there are $n(B)$ elements that are in B but not in A. Therefore,

$$n(A \cup B) = n(A) + n(B) \tag{5.3.5}$$

□

Lemma 5.3.1. *If A and B are finite sets, and A and B are not disjoint, then $A \cup B$ and $A \cap B$ are finite. Thus,*

$$n(A \cup B) = n(A) + n(B) - n(A \cap B) \tag{5.3.6}$$

Corollary 5.3.1. *If A, B, and C are finite sets and A, B and C are not disjoint, then their union and intersection are finite, Thus,*

$$\begin{aligned} n(A \cup B \cup C) &= n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) \\ &\quad -n(B \cap C) + n(A \cap B \cap C) \end{aligned} \tag{5.3.7}$$

Example 5.3.1

In a class of 60 students, 36 offer Mathematics and 28 offer physics. If 8 students do not offer any of the two subjects,

(i) How many students offer both subjects;

(ii) How many students offer Mathematics only;

(iii) How many students offer physics only;

(iv) How many students offer only one subject.

Solution

Let M denote Mathematics, & P denote Physics;
then $n(M)$= number of students that offer Mathematics,
and $n(P)$ = number of students that offer physics.
Thus,

$$\begin{aligned} n(M \cup P) &= \text{number of students that offer either Mathematics} \\ &\quad \text{or Physics or both} \end{aligned}$$

Similarly,

$$n(M \cap P) = \text{number of students that offer both subjects}$$

The Venn diagram is thus displayed in Figure 5.6 below.

Since 8 students do not offer any of the two subjects, it follows that;

$$n(M \cup P) = U - n(M \cup P)' = 60 - 8 = 52.$$

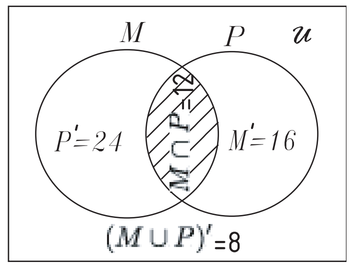

Figure 5.6:

But,

$$n(M \cup P) = n(M) + n(P) - n(M \cap P)$$

(Since M and P are not disjoint - Lemma 5.3.1.
i.e.

$$\begin{aligned} 52 &= 36 + 28 - n(M \cap P) \\ &= 64 - n(M \cap P) \\ \therefore \quad n(M \cap P) &= 64 - 52 = 12 \end{aligned}$$

(i) Number of students that offer both subjects is,

$$n(M \cap P) = 12$$

(ii) Number of student that offer mathematics only i.e.,

$$n(P') = n(M \cup P) - n(P)$$

Since P and P' are disjoint,

$$\begin{aligned} n(P') &= n(M) - n(M \cap P) \\ &= 52 - 28 = 24 \\ \text{or} \quad n(P') &= 36 - 12 = 24 \end{aligned}$$

(iii) Number of students that offer physics only;

$$\begin{aligned} n(M') &= n(M \cup P) - n(M) \\ &= 52 - 36 = 16 \\ \text{or} \quad n(M') &= n(P) - n(M \cap P) \\ &= 28 - 12 = 16 \end{aligned}$$

(iv) Number of students that offer only one subject;

$$\begin{aligned} n(P' \cup M') &= n(P') + n(M') \quad \textit{(Theorem 5.3.2)} \\ &= 24 + 16 = 40 \end{aligned}$$

Example 5.3.2

In a survey of 120 people, it was found that 50 people read Daily Times, 52 read The Guardian, and 52 read Vanguard. Also 18 read both Daily Times and Vanguard, 16 read both Guardian and Vanguard, and 22 read both Daily Times and Guardian, and 16 read none of the three newspapers.

(i) Find the number of people who read all three newspapers,

(ii) Determine the number of people who read exactly two newspapers,

(iii) Find the number of people who read only one newspaper.

Solution

Let T, G and V denotes the sets of people that read the Daily Times, The Guardian, and Vanguard newspapers respectively.
Then, we have

$$\begin{aligned} n(U) &= 120; \quad n(G) &= 52 \\ n(T) &= 50; \quad n(V) &= 52 \\ n(T \cap G) &= 22, \quad n(G \cap V) &= 16 \\ n(T \cap V) &= 18, \quad n(T \cup G \cup V)' &= 16 \end{aligned}$$

The Venn diagram is displayed in Figure 5.7 below.

(i) Since 16 people read none of the three newspapers, thus

$$\begin{aligned} n(T \cup G \cup V) &= n(U) - n(T \cup G \cup V)' \\ &= 120 - 16 \\ &= 104 \end{aligned}$$

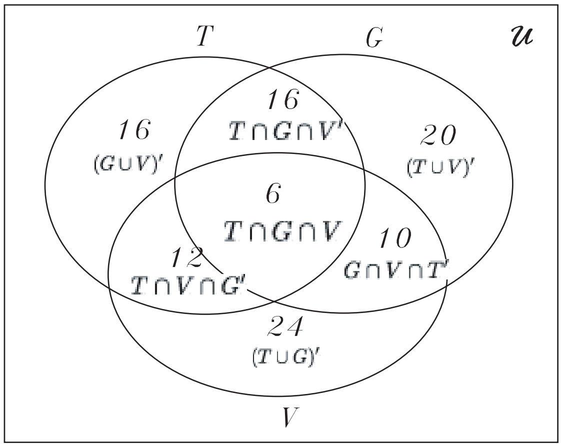

Figure 5.7:

By corollary 5.3.1

$$\begin{aligned} n(T \cup G \cup V) &= n(T) + n(G) + n(V) - n(T \cap G) \\ &\quad -n(G \cap V) - n(T \cap V) \\ &\quad +n(T \cap G \cap V) \\ \text{i.e.} \quad 104 &= 50 + 52 + 52 - 22 - 16 - 18 \\ &\quad +n(T \cap G \cap V) \\ &= 98 + n(T \cap G \cap V) \\ \text{i.e.} \quad n(T \cap G \cap V) &= 6 \end{aligned}$$

Therefore 6 people read all three newspapers.

(ii) (a) Those that read Daily Times and Guardian only are:

$$\begin{aligned} n(T \cap G \cap V') &= n(T \cap G) - n(T \cap G \cap V) \\ &= 22 - 6 \\ &= 16 \end{aligned}$$

read Daily Times and The Guardian only.

(b) Those that read Guardian and Vanguard only are:

$$\begin{aligned} n(G \cap V \cap T') &= n(G \cap V) - n(T \cap G \cap V) \\ &= 16 - 6 \\ &= 10 \end{aligned}$$

read Guardian and Vanguard only.

(c) Those that read Daily Times and Vanguard only are:

$$\begin{aligned} n(T \cap V \cap G') &= n(T \cap V) - n(T \cap G \cap V) \\ &= 18 - 6 \\ &= 12 \end{aligned}$$

read Daily Times and Vanguard only.

$\therefore \quad 16 + 10 + 12 = 38$ read exactly two newspapers.

(iii) (a) Those that read only Daily Times are:

$$\begin{aligned} n(G \cup V)' &= n(T \cup G \cup V) - n(G \cup V) \\ &= n(T \cup G \cup V) - [n(G) + n(V) - n(G \cap V)] \\ &= 104 - (52 + 52 - 16) \\ &= 16 \quad \text{read only Daily Times.} \end{aligned}$$

(b) Those that read only Guardian are:

$$\begin{aligned} n(T \cup V)' &= n(T \cup G \cup V) - n(T \cup V) \\ &= n(T \cup G \cup V) - [n(T) + n(V) - n(T \cap V)] \\ &= 104 - (50 + 52 - 18) \\ &= 20 \quad \text{read only Guardian.} \end{aligned}$$

(c) Therefore, those that read only one newspaper are:

$$\begin{aligned} n(T \cup G)' &= n(T \cup G \cup V) - n(T \cup G) \\ &= n(T \cup G \cup V) - [n(T) \\ &\quad +n(G) - n(T \cap G)] \\ &= 104 - (50 + 52 - 22) \\ &= 24 \quad \text{read only Vanguard.} \\ \therefore \quad 16 + 20 + 24 &= 60 \quad \text{read only one newspaper.} \end{aligned}$$

Alternatively, we can as well determine the number of people that read exactly one newspaper using the Venn diagram above as follows:

(a) Those that read only Daily Times are:

$$\begin{aligned} n(G \cup V)' &= n(T) - n(T \cap G \cap V') - n(T \cap V \cap G') \\ &\quad -n(T \cap G \cap V) \\ &= 50 - 16 - 12 - 6 \\ &= 16 \quad \text{(only Daily Times)} \end{aligned}$$

(b) Those that read only Guardian are:

$$\begin{aligned} n(T \cup V)' &= n(G) - n(T \cap G \cap V') - n(G \cap V \cap T') \\ &\quad -n(T \cap G \cap V) \\ &= 52 - 16 - 10 - 6 \\ &= 20 \quad \text{(only Guardian)} \end{aligned}$$

(c) Those that read only Vanguard are:

$$\begin{aligned} n(T \cup G)' &= n(V) - n(T \cap V \cap G') - n(G \cap V \cap T') \\ &\quad - \cap (T \cap G \cap V) \\ &= 52 - 12 - 10 - 6 \\ &= 24 \quad \text{(only Vanguard)} \end{aligned}$$

$\therefore$ we have that $16 + 20 + 24 = 60$ people read only one newspaper.

5.4 Ordered Pairs, Product Sets

Definition 3 *An ordered pair can be defined intuitively as two elements, a, b, such that one, e.g. a is designated as the first element and the other as the second element. Such an ordered pair is written (a, b), with the understanding that two pairs (a, b) and (c, d) are equal if and only if $a = c$ and $b = d$. Generally, if an ordered n-turple $(a_1, a_2, \ldots, a_n) = (a'_1, a'_2, \ldots, a'_n)$,*
thus, $a_1 = a'_1, a_2 = a'_2, \ldots, a_n = a'_n$.

Definition 4 *The Cartesian Product or Product Set of two arbitrary sets, say A and B, is the set of all ordered pairs (a, b) such that $a \in A$ and $b \in B$, which is written as $A \times B$, and is read 'A cross B'.*
i.e.,

$$A \times B = \{(a, b) : a \in A, b \in B\}$$

In fact, for any finite sets A and B,

$$n(A \times B) = n(A) \cdot n(B)$$

In particular, if A or B is a null set, then so likewise is $A \times B$. If A has m elements and B has n elements; $A \times B$ has mn elements.

Generally, for any finite number of sets $A_1, A_2, \ldots, A_n$, the set of all ordered n-turples $(a_1, a_2, \ldots, a_n)$, where $a_1 \in A_1, a_2 \in A_2, \ldots, \quad a_n \in A_n$, is called the product of the sets $A_1, A_2, \ldots, A_n$, and is denoted by,

$$A_1 \times A_2 \times \ldots \times A_n \quad \text{or} \quad \prod_{i=1}^{n} A_i$$

One often writes A^n instead of $A \times A \times \ldots \times A$, where there are n factors all equal to A.

Example 5.4.1

(i) The Cartesian plane is the set of all ordered pairs (x, y); $(x, y) \in \mathbb{R}^2$, where $x, y \in \mathbb{R}$ and $\mathbb{R}$ denotes the set of real numbers.

(ii) Let $A = (3, 4, 5), \quad B = \{a, b\}$ then $A \times B = \{(3, a), (3, b), (4, a), (4, b), (5, a), (5, b)\}$.

(iii) $\mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R} = \{(E_1, E_2, E_3) : E_1, E_2, E_3 \in \mathbb{R}\}$ denotes the usual Euclidean three - dimensional space.

5.5 Relations

Definition 5 *Let A and B be two arbitrary sets. Indeed, one can define a relation $\Re$, from A to B as a set of ordered pairs (a, b) in*

$A \times B$. We write $a\Re b$, (to be read a is in relation $\Re$ to b), otherwise, a is not related to b. In particular, a relation from a set A to the same set A is called a relation on A.

Example 5.5.1

(i) Marriage is a relation from the set M of men to the set W of women. For instance, given any man $m \in M$ and any woman, $w \in W$, either m is married to w or m is not married to w.

(ii) Consider $\mathbb{R} = (a, b);\ a, b \in \mathbb{Z}$ where (a, b) means a divides b commonly written as a/b is a relation.

(iii) Consider also, $\mathbb{R} = (\mathbb{R} \times \mathbb{R}, f(x, y))$, where $f(x, y)$ means $y = x^2$, $\mathbb{R}$ is a relation. $\mathbb{R}^2$ can be represented by the set of points in the plane as in Figure 5.8.

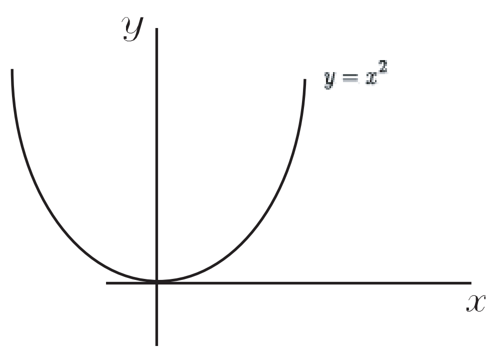

Figure 5.8: Set of points in a plane

Then $\mathbb{R}^* = \{(x, x^2) : x \in \mathbb{R}\}$ and the points of R are given in fig.5.8 above.

Remark 9 *In order to distinguish a relation from a set of ordered pair, we denote the relation by $\mathbb{R}$, then $\mathbb{R}^*$ denotes the set of ordered pairs, called the graph of $\mathbb{R}$.*

Definition 6 *Let $\mathbb{R}$ be a relation from a set A to a set B. The domain D of a relation $\mathbb{R}$, is the subset of A consisting of first* *co-ordinate elements of* $\mathbb{R}^*$.

$$i.e., \ D = \{x : (x, y) \in \mathbb{R}^* \text{ for some } y\}.$$

The range G *of* $\mathbb{R}$ *is the subset of* B *consisting of the second co-ordinate elements of* $\mathbb{R}^*$ *i.e.*

$$G = \{y : (x, y) \in \mathbb{R}^* \text{ for some } x\}$$

Example 5.5.2

Consider,

$$\begin{aligned} A &= \{1, 2, 3, 4\} \\ B &= \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\} \end{aligned}$$

Let $\mathbb{R} = (A, B, f(x, y))$, where $f(x, y)$ means $y = 3x$. Then,

$$\mathbb{R}^* = \{(1, 3), (2.6), (3, 9)\}$$

from the graph $\mathbb{R}^*$ of the co-ordinate points.

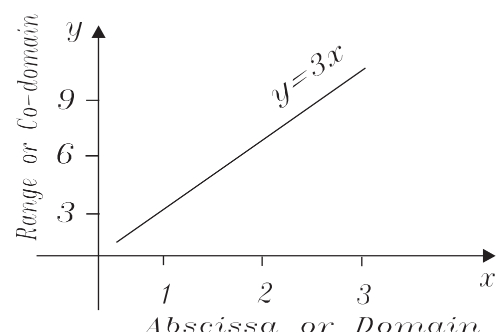

Figure 5.9:

$$\begin{aligned} \text{Domain }, D &= \{1, 2, 3\} \\ \text{Range}, G &= \{3, 6, 9\} \end{aligned}$$

Definition 7 *Let* $\mathbb{R} = (A, B, f(x_1, y))$ *be a relation. The inverse of* $\mathbb{R}$*, denoted by* $\mathbb{R}^{-1}$ *is a relation from* B *to* A *which consists of the ordered pair which when reversed belong to* $\mathbb{R}$ *and whose graph is given by*

$$(\mathbb{R}^{-1})^* = \{(y, x) : (x, y) \in \mathbb{R}^*\}$$

Example 5.5.3

In Example 5.5.2,

$$(\mathbb{R}^{-1})^* \text{ is given by } \{(3,1),(6,2),(9,3)\}$$

Clearly, if $\mathbb{R}$ is any relation, then $(\mathbb{R}^{-1})^{-1} = \mathbb{R}$. Also the domain of $\mathbb{R}^{-1}$ equals the range of $\mathbb{R}$, and conversely, the range of $\mathbb{R}^{-1}$ equals the domain of $\mathbb{R}$.

Definition 8 *If f and g are relations such that f and g are both subsets of $A \times B$. We define the composition (or product) of f and g as the relation,*

$$g \cdot f = \{(x,z) : (x,y) \in f, (y,z) \in g \text{ for some } y\}$$

Example 5.5.4

Let $f = \{(1,2)\}, \quad g = \{0,1)\}$.
Then $f \cdot g = \{(0,2)\}$.
But $g \cdot f = \emptyset$
Thus, $f \cdot g \neq g \cdot f$, which shows that composition of relations may not be commutative.

5.5.1 Equivalence Relations

Definition 9 *Let A be a set. A relation $\Re$ in A is said to be,*

(i) reflexive if $a\Re a$ for all $a \in A$. i.e., $(a,a) \in \mathbb{R}^*$ for all $a \in A$;

(ii) Symmetric if $a\Re b$ implies that $b\Re a$ for all $a, b \in A$. i.e., $(a,b) \in \mathbb{R}^*$ implies that $(b,a) \in \mathbb{R}^*$ for all $a, b \in A$.

(iii) transitive if $a\Re b$, and $b\Re c$ imply that $a\Re c$ i.e., $(a,b) \in \mathbb{R}^*$, $(b,c) \in \mathbb{R}^*$ imply that $(a,c) \in \mathbb{R}^*$.

Thus for any set A, an *equivalence relation* on A is a relation $\mathbb{R}^* \subset A \times A$ satisfying (i), (ii), and (iii) properties above.

Example 5.5.5

(i) The relation $\subset$ (proper subset) of a set, i.e. inclusion is not an equivalence relation, since it is reflexive, and transitive, but not symmetric because $A \subset B$ does not imply $B \subset A$.

(ii) The relation of 'equality'(=) in $\mathbb{Z}$ (set of integers), is an equivalence relation since it satisfies reflexive, transitive and symmetric properties.

5.6 Mappings or Functions

Literally, we could say that a function is something closely related to or dependent on something else. But in mathematics, it is a rule of correspondence between 2 sets such that there is a unique element in 1 set assigned to each element in the other. A proper understanding of Relations just discussed in our previous section will be very helpful in appreciating this topic.

5.6.1 Relations and Function in the Plane

Let x and y denote real numbers, so that the ordered pair (x, y) can be thought of as representing rectangular coordinates of points in the xy-plane (or a complex number). We frequently encounter such expressions as,

$$xy = 1, \quad x^2 + y^2 = 1, \quad \text{or} \quad x^2 + y^2 \leq 1, \quad x < y$$

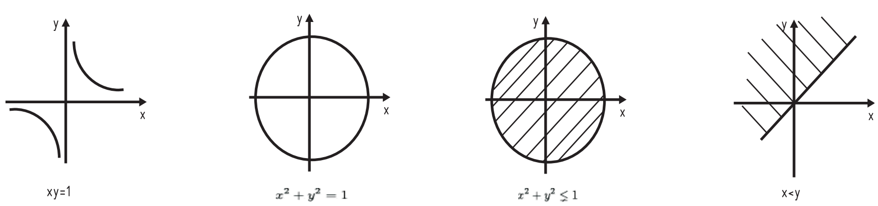

Figure 5.10: Set of ordered pairs (x, y) of real numbers

Each of these expressions defines a certain set of ordered pairs (x, y) of real numbers, namely, the set of all pairs (x, y) for which the expression is satisfied. Such a set of ordered pairs is called a *plane relation*. The corresponding set of points plotted in the xy-plane is called the graph of the relations as described above.

Every set of ordered pairs (x, y) constitutes a plane relation. Certain special kinds of relation are known as *functions*. A *function* is a relation whose graph has the property that every vertical line intersects it at most once. In other words, whenever two points (x, y_1), and (x, y_2) in the graph have the same x-coordinate, then they must also have the same y-coordinate; i.e., they must have $y_1 = y_2$. Thus, for each point (x, y) of the graph of a function, the x-coordinate uniquely determines the y-coordinate. For example, the relation defined by the equation $xy = 1$; is a function because for each x there is exactly one y such that $xy = 1$, namely, $y = \frac{1}{x}$. On the other hand, the relation defined by the equation $x^2 + y^2 = 1$ is not a function because there are points of the graph which have the same x-coordinates but different y-coordinates. Two such points are $(0, 1)$ and $(0, -1)$. Similarly the other examples given above are not functions.

If F is a set of ordered pairs which constitutes a function, then instead of writing $(x, y) \in F$ to indicate the pair (x, y) in the set F, we write $y = F(x)$. This is consistent with the fact that for each pair (x, y) in F, the second member y is uniquely determined by x.

If each point on the graph of a function F is projected vertically on the x-axis, we obtain a set of points on the x-axis known as the *domain* of the function, denoted by $D(F)$. Similarly, projecting the graph horizontally on the y-axis gives us another set called the *co-domain* or *range* of the function, denoted by $R(F)$.

Definition 10 *A relation F is called a function provided that $(x, y) \in F$ and $(x, z) \in F$ implies $y = z$.*

In other words, a function is a set of ordered pairs which has the special property that whenever two pairs (x, y) and (x, z) in the set have the same first element, they must also have the same second element.

Example 5.6.1

Consider the function F defined as follows;

$$F = \{(x, y) : y = x^3, \text{ if } -1 \leq x \leq 1\}$$

The graph of F is illustrated as follows:

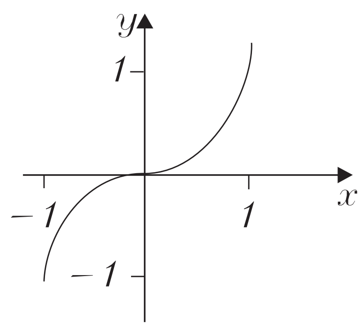

Figure 5.11:

Here, the domain $D(F)$ consist of the interval $-1 \leq y \leq +1$, while the co-domain or the range $R(F)$ consists of the interval $-1 \leq x \leq +1$.

Generally speaking, whenever a function is given by means of an equation of the form

$$y = F(x) \tag{5.6.1}$$

we say that y is a function of x, and we call x the *independent variable* and y the *dependent variable*.

Remark 10 *The terms transformation, operator, correspondence, and mapping are used as synonymous of 'function'.*

5.6.2 One-to-One and Onto Mapping

Let A be a subset of the domain of F, we say that F is defined on A. In this case, the set $F(x)$ such that $x \in A$ is called the image of A under F and is denoted by $F(A)$. If $F(A) \subset B$, then F is called a mapping of A into B. Alternatively, we say that F is a mapping of A into B and write it as

$$F : A \mapsto B.$$

A relation associates an element of one set say A, (the domain) with one or more elements of another set B (the co-domain), which may be the same set as the first. We refer to those elements in the second set as the images of the element in the domain (A) and the set of all the images are called the range.

When the domain consists of real numbers, F is called a function of one real variable. However, if the domain of F is a set of points in the plane (or a set of complex numbers), then F is called a function of two variables (or a function of a complex variable).

5.6.3 One-to-One Mapping

Let F be a function defined on a set A. We say F is a *One-to-one* or *injective* on A if, and only if, for every $x, y \in A$,

$$F(x) = F(y) \text{ implies } x = y$$

In other words, a relation is said to be one-to-one, if each element in the domain has only one image in the range. This is the same as saying that a function which is one-to-one on A assigns distinct function values to distinct members of A

Now suppose $(a, b, c) \in A$ and $(x, y, z) \in B$, then a mapping $F : A \mapsto B$ given as;

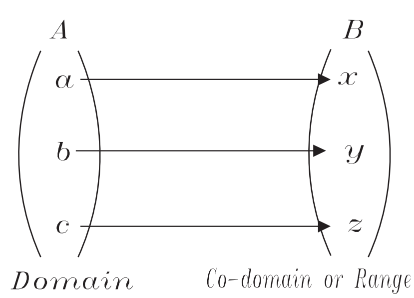

Figure 5.12: One-to-one mapping

is called a one-to-one mapping. The set $(a, b, c) \in A$ is called the domain and the elements $(x, y, z) \in B$ are called the images or co-domain of A.

Example 5.6.2

A relation is defined with the domain, X= $\{-2, 2, 4\}$ and co-domain $\mathbb{R}$. Find the range in each case when the relation,

(i) $F(x) = \frac{1}{2}x$

(ii) $F(x) = 2x + 1$

Solution

(i) $F : x \mapsto \frac{1}{2}x$ i.e.;

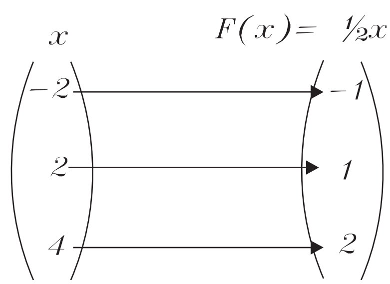

Figure 5.13:

(ii) $F : x \mapsto 2x + 1$ i.e.;

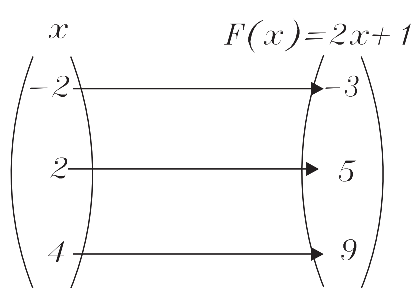

Figure 5.14:

Example 5.6.3

Find the range defined over the domain $\{x : 1 \leq x \leq 4\}$, where

$$F : x \mapsto \frac{1}{2x - 1}$$

Solution

$$F(1) = \frac{1}{2(1) - 1} = 1; \quad F(4) = \frac{1}{2(2) - 1} = \frac{1}{7}$$

$\therefore$ the range is the set $\{y : \frac{1}{7} \leq y \leq 1\}$

5.6.4 Onto Mapping

A mapping $F : A \mapsto B$ is said to be *onto* or *surjective* if the range of F is equal to B, i.e. given any $y \in B$, there exists $x \in A$ such that $F(x) = y$. In other words, it is not necessary for one and only one element in the domain to correspond to one and only one element in the range as illustrated in the figure below:

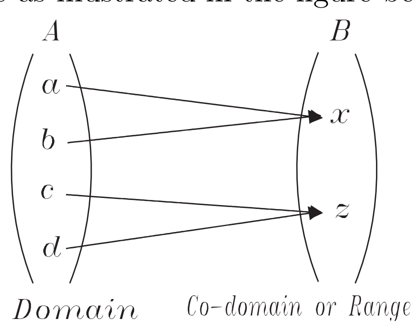

Figure 5.15: Onto mapping

Example 5.6.4

Find the range in the function $F : x \mapsto x^2$, defined over the domain $D = \{-1, -2, 1, 2\}$.

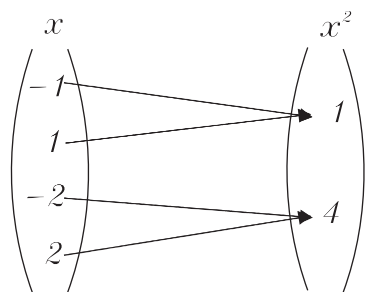

Figure 5.16:

Solution

Observe that the function $F : x \mapsto x^2$ above in Figure 5.16 maps two elements in the domain into one element of the range (but it would be one-to-one if the domain was restricted to only positive integers).

Remark 11 *The important property of a function is that for every element in the domain there is one and only one image in the range, so that the relation $F : x \mapsto \sqrt{x}$ is not a function if the range is the set $\mathbb{R}$, but is a function if the range is restricted to $\mathbb{R}^+$.*

Definition 11 *A mapping $F : A \mapsto B$ is said to be bijective if it is injective and surjective.*

Definition 12 *A mapping $F : A \mapsto A$ is called an identity mapping if $F(x) = x$ for all $x \in A$, referred to a mapping of A into itself. The identity mapping is a bijective mapping.*

5.6.5 Inverse Functions

Recall that a one-to-one mapping maps every element in the domain onto a unique image in the range; we can extend this to find which element has been mapped into any given image in the range. If we denote the image of an element A in the domain by B, then let $x \in A$ and $y \in B$, such that,

$$F : A \mapsto B$$

then,

$$\begin{aligned} y &= F(x) \\ \text{i.e.} \quad x &= F(y) \end{aligned}$$

So that the function $F : B \mapsto A$ maps any element in the range back into its original element in the domain. Such a function is called the *inverse function*, and is usually denoted by F^{-1}, so that if $F : A \mapsto B, \quad F^{-1} : B \mapsto A$.
i.e.;

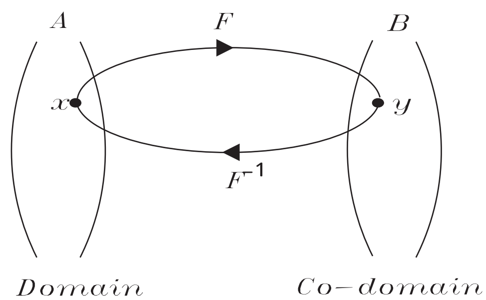

Figure 5.17: Inverse mapping

Thus, if F as well as F^{-1} are single-valued, then F is a one-to-one mapping.

Example 5.6.5

Find the inverse of the function,

(i) $F : x \mapsto 3x - 1$, defined over the domain $D = \{-1, 0, 1, 2\}$

(ii) $F : x \mapsto \sin x^{\circ}, \quad \{x : -90 \leq x \leq 90\}$

Solution

(i) If we take any element in the domain say -1, then,

$$y = 3x - 1 = 3(-1) - 1 = -4 \quad \text{and} \quad x = \frac{1}{3}(y + 1)$$

i.e.,

$$F : x \mapsto 3x - 1; \quad F^{-1} : y \mapsto \frac{1}{3}(y + 1)$$

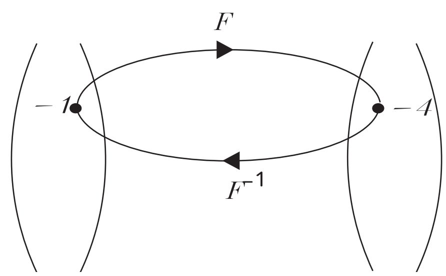

Figure 5.18:

(ii) If we take any element in the domain say 30, then F maps 30 into $\sin 30°$ i.e., $\frac{1}{2}$ as follows:

$$\begin{aligned} y &= \sin x° \\ &= \sin 30° \\ &= \frac{1}{2} \\ \text{but} \quad x &= \sin^{-1} y \end{aligned}$$

Hence,

$$F : x \mapsto \sin x°, \quad F^{-1} : y \mapsto \sin^{-1} y$$

i.e.,

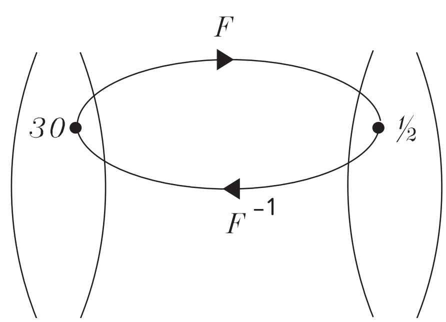

Figure 5.19:

5.6.6 Composite Functions

Let F and G be any two functions such that $R(F) \subset D(G)$, we can form a new function , the composite GF of F and G, defined as follows: for every x in the domain of F,

$$GF(x) = G[F(x)].$$

Since $R(F) \subset D(G)$, the element $F(x)$ is in the domain G, and therefore it is justified to consider $G[F(x)]$. In general, it is not true that $GF = FG$. In fact, FG may be meaningless unless the range of G is contained in the domain of F. In particular, two mappings F and G i.e. $F, G : A \mapsto B$ are equal if,

$$F(x) = G(x), \quad \text{for all} \quad x \in A$$

Example 5.6.6

If $F : x \mapsto \frac{1}{2}x$ and $G : x \mapsto x - 2$, find,

(i) $GF(2)$ and $FG(2)$

(ii) $G^{-1}F^{-1}(0)$ and $F^{-1}G^{-1}(0)$

(iii) $G^{-1}F^{-1}(1)$ and $F^{-1}G^{-1}(1)$

Solution

(i)

$$\begin{aligned} F(x) &= \frac{1}{2}x \\ \therefore \qquad F(2) &= 1 \end{aligned}$$

$$\begin{aligned} G(x) &= x - 2 \\ \therefore \qquad GF(2) &= G(1) \\ &= -1 \end{aligned}$$

$$\begin{aligned} G(2) &= 2 - 2 \\ &= 0 \\ \therefore \qquad FG(2) &= F(0) = -2 \end{aligned}$$

(ii) $F^{-1} = 2y$ and $G^{-1} = y + 2$;
Thus,

$$\begin{aligned} F^{-1}(0) &= 0 \\ \therefore \quad G^{-1}F^{-1}(0) &= G^{-1}(0) \\ &= 2 \end{aligned}$$

Similarly,

$$\begin{aligned} G^{-1}(0) &= 2 \\ \therefore \quad F^{-1}G^{-1}(0) &= F^{-1}(2) \\ &= 4 \end{aligned}$$

(iii)

$$\begin{aligned} F^{-1}(1) &= 2 \\ \therefore \quad G^{-1}F^{-1}(1) &= G^{-1}(2) \\ &= 4 \end{aligned}$$

Similarly,

$$\begin{aligned} G^{-1}(1) &= 3 \\ \therefore \quad F^{-1}G^{-1}(1) &= F^{-1}(3) \\ &= 6 \end{aligned}$$

Example 5.6.7

Let $F : A \mapsto B$ and $G : B \mapsto C$, be any two injective mappings. Prove that their composite $GF : A \mapsto C$ is also injective.

Solution

Let $GF(x_1) = GF(x_2)$ in C, we must show that $x_1 = x_2$. Consequently, by definition of composites,

$$G[F(x_1)] = G[F(x_2)]$$

Since G is one-to-one, it follows that $F(x_1) = F(x_2)$ implies $x_1 = x_2$. Therefore GF is injective or one-to-one.

Chapter 6

Real Sequences and Series, Theory of Mathematical Induction

Sequences and series are introduction to an important part in mathematics and its many applications can be found in the sciences, engineering and in the mathematics of finance. Formulae for the computation of interest, hire purchase, present values, mortgage loans etc., are derived using series. Similarly, many numerical methods are based on the theory of series; for instance, the construction of tables of logarithms and trigonometric functions and the calculation of important constants such as e and π are best accomplished with the help of series.

6.1 Sequences

By sequence, we mean the order in which things or events occur or are arranged; sequence is synonymous with the word 'order'. A sequence of numbers is a set of numbers (called terms) arranged in a definite order. The successive terms of the sequence is derived from its preceding term according to some rule. A sequence is usually written as;

$$T_1, T_2, T_3, \ldots, T_n, \ldots$$

where T_1 = first term, $T_n = n^{th}$ term (also called the general term of the sequence), while the subscript of each term represents the term number or position.

6.1.1 Finite and Infinite Sequences

If the number of terms of a sequence is countable (i.e. the counting has a limit), then the sequence is called *a finite sequence*. The last term of a finite sequence is represented by the symbol T_n or ℓ. For example

(i) $-5, -3, -1, 1, 3, 5, 7.$

(ii) $2, 4, 8, 16, 32, 64.$

The set of the numbers in (i) is a sequence since each term is obtained by adding 2 to the preceding one. Here the first term, $T_1 = -5$ and the last term, $T_n = 7$. Similarly in (ii) each term of the sequence is obtained by multiplying the predecessor by 2 and it has its last term as 64. Observe that in both cases the number of terms is finite hence the name finite sequence.

Conversely, if the number of terms in a sequence is uncountable (i.e. the counting has no limit), then the sequence is called an infinite sequence. The symbol T_n represents the n^{th} term of the sequence. For example

$$1, 2, 3, 4, \ldots$$

(i.e. the set of natural numbers) is an infinite sequence and the proceeding dots indicate that the sequence is infinite.

6.1.2 Sequence as a Function

Observe that from the sequences discussed so far, it is possible to find out each succeeding term by inspection, however, this is not always the case with many sequences. In most cases, each term of a sequence is a function of n (i.e. n being a natural number) such that

$$T_n = f(n) \tag{6.1.1}$$

Example 6.1.1

Write down the first four terms of the sequence whose n^{th} terms are given by;

$$\text{(i)}\quad n^2; \quad \text{(ii)}\quad 2^{n-1}; \quad \text{(iii)}\quad \frac{n}{n+1}$$

Solution

(i) When,

$$\begin{array}{llll} n=1, & T_1 = & 1^2 = & 1 \\ n=2, & T_2 = & 2^2 = & 4 \\ n=3, & T_3 = & 3^2 = & 9 \\ n=4, & T_4 = & 4^2 = & 16 \end{array}$$

$\therefore$ the sequence is $1, 4, 9, 16, \ldots$

(ii)

$$\begin{array}{lllll} T_1 & = & 2^{1-1} & = & 1 \\ T_2 & = & 2^{2-1} & = & 2 \\ T_3 & = & 2^{3-1} & = & 4 \\ T_4 & = & 2^{4-1} & = & 8 \end{array}$$

$\therefore$ the sequence is $1, 2, 4, 8, \ldots$

(iii)

$$\begin{array}{lll} T_1 & = & \frac{1}{1+1} = \frac{1}{2} \\ T_2 & = & \frac{2}{2+1} = \frac{2}{3} \\ T_3 & = & \frac{3}{3+1} = \frac{3}{4} \\ T_4 & = & \frac{4}{4+1} = \frac{4}{5} \end{array}$$

$\therefore$ the sequence is $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots$

Example 6.1.2

Find the formula for the n^{th} terms of the sequence given by;

(i) $-4, -1, 2, 5$

(ii) $1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \frac{1}{25}, \ldots$

(iii) $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$

Solution

(i) By inspection, T_2 can be obtained by
$T_2 = T_1 + 3 = -4 + 3 = -1$.
Similarly,

$$\begin{aligned} T_3 &= T_2 + 3 \\ &= -1 + 3 = 2 \end{aligned}$$

and,

$$\begin{aligned} T_4 &= T_3 + 3 \\ &= 2 + 3 = 5 \end{aligned}$$

$\therefore$ the n^{th} term is given by;

$$T_n = -4 + 3(n-1) = 3n - 7$$

(ii)

$$\begin{aligned} T_1 &= 1 = \frac{1}{1^2} \\ T_2 &= \frac{1}{4} = \frac{1}{2^2} \\ T_3 &= \frac{1}{9} = \frac{1}{3^2} \\ T_4 &= \frac{1}{16} = \frac{1}{4^2} \\ T_5 &= \frac{1}{25} = \frac{1}{5^2} \\ \therefore \quad T_n &= \frac{1}{n^2} \end{aligned}$$

(iii)

$$\begin{aligned} T_1 &= \frac{1}{2} = \frac{1}{1+1} \\ T_2 &= \frac{1}{3} = \frac{1}{2+1} \\ T_3 &= \frac{1}{4} = \frac{1}{3+1} \\ T_4 &= \frac{1}{5} = \frac{1}{4+1} \\ \therefore \quad T_n &= \frac{1}{n+1} \end{aligned}$$

Example 6.1.3

(i) If $T_n = (n+1)(n+2)$, which term of the sequence $T_1, T_2, \ldots,$ is 20.

(ii) In a sequence given by $T_n = a + bn$, the 6^{th} and 11^{th} terms are 0 and 10 respectively. Find the values of a and b, and hence the n^{th} term.

Solution

(i) Let $T_n = 20$,
Then,

$$\begin{aligned} (n+1)(n+2) &= 20 \\ \text{i.e.,} \quad n^2 + 3n - 18 &= 0 \\ (n+6)(n-3) &= 0 \\ \therefore \quad n &= -6 \quad \text{or} \quad 3 \end{aligned}$$

But n cannot be negative, therefore, $n = 3$, hence $T_3 = 20$.

(ii) If $T_n = a + bn$
Then,

$$\begin{aligned} T_6 &= a + 6b = 0 \\ \text{and} \quad T_{11} &= a + 11b = 10 \end{aligned}$$

Solving simultaneously, we have $a = -12, \quad b = 2.$

$$\therefore \qquad T_n \;=\; 2n - 12$$

6.2 Series

A series is the sum where each term of a sequence is added to the previous one. It is obtained by forming the sum of the terms of a sequence. A given series can be finite or infinite, according to whether the number of terms of the sequence is finite or infinite. A series is usually written as the sum of the n terms of a sequence and is generally denoted by;

$$S_n = T_1 + T_2 + T_3 + \ldots + T_n + \ldots = \sum_{r=1}^{\infty} T_r \qquad (6.2.1)$$

where the dots indicate that the series is infinite. The sum of a finite number of consecutive terms of a series, beginning with the first term is referred to as a *partial sum* of the series, and is given as

$$\begin{array}{lcll} S_1 & = & T_1 & : \text{first partial sum} \\ S_2 & = & T_1 + T_2 & : \text{second partial sum} \\ \vdots & & & \\ S_n & = & T_1 + T_2 + T_3 \;\; + \ldots + T_n : & n^{th} \text{ partial sum} \\ & = & \sum_{r=1}^{n} T_r & \end{array} \qquad (6.2.2)$$

Example 6.2.1

Let $S_n = n(n+2)$ be the sum of the first n terms of the sequence. Find the expression for the r^{th} and the $(n+1)^{th}$ terms of the sequence, and hence find the first four terms of the sequence.

Solution

If T_r denotes the r^{th} term, then T_r is the difference between the sum of the first r terms and the sum of the first $(r-1)$ terms i.e.,

$$T_r = S_r - S_{r-1} \qquad (6.2.3)$$

Thus,

$$\begin{aligned} T_r &= r(r+2) - [(r-1)(r-1+2)] \\ &= r^2 + 2r - [(r-1)(r+1)] \\ &= r^2 + 2r - r^2 + 1 \\ &= 2r + 1 \end{aligned}$$

and,

$$\begin{aligned} T_{n+1} &= (n+1)[(n+1)+2] \\ &= (n+1)(n+3) \\ &= n^2 + 4n + 3 \end{aligned}$$

Hence, when $r = 1$;

$$\begin{aligned} T_1 &= 2(1) + 1 = 3 \\ r = 2, \quad T_2 &= 2(2) + 1 = 5 \\ r = 3, \quad T_3 &= 2(3) + 1 = 7 \\ r = 4, \quad T_4 &= 2(4) + 1 = 9 \end{aligned}$$

$\therefore$ the first four terms is given as $3, 5, 7, 9$.

6.3 Arithmetic Progression (A.P.)

Over the years from our elementary school, we have been very familiar with the subject – Arithmetic. In retrospect therefore, arithmetic as we have already known is the mathematics of integers under addition, subtraction, multiplication and division. However, in this section, we shall be concerned with the application of this arithmetic laws as discussed in Chapter 1 to the arrangement of rational numbers – here referred to as terms, in ascending or descending order to form what we may refer to as Sequences or Series.

6.3.1 Arithmetic Sequences

An arithmetic sequence is a sequence of terms that increase or decreases by a constant value. The constant value is known as the *common difference* of the sequence and is usually denoted by d. For example,

(i) $-2, 0, 2, 4, 6, \ldots$

(ii) $5, 2, -1, -4, \ldots$

In (i), the common difference is 2 while the common difference in (ii) is -3.

Theorem 6.3.1. *To find a formula for the n^{th} term of an A.P., with common difference given as d and first term as a.*

Proof. Starting from the given first term, we have;

$$
\begin{aligned}
T_1 &= a \\
T_2 &= T_1 + d = a + d \\
T_3 &= T_2 + d = (a + d) + d = a + 2d \\
T_4 &= T_3 + d = (a + 2d) + d = a + 3d \\
&\vdots \\
T_r &= T_{r-1} + d = [a + (r - 2)d] + d = a + (r - 1)d \\
&\vdots
\end{aligned}
$$

Hence, if we now replace the r^{th} term in the last equation with n, we then have;

$$T_n = a + (n - 1)d \qquad (6.3.1)$$

which completes the proof. □

Example 6.3.1

Determine whether each of the sequence;

(i) $9, 4, -1, -6, \ldots$

(ii) $-x, x + 3, 3x + 6, \ldots$

is an *A.P.*, and hence find the 12^{th} term for both cases. Determine the formula for the n^{th} term of (i) and (ii) respectively.

Solution

(i) Here, $T_2 - T_1 = 4 - 9 = -5 \quad \text{and} \quad T_3 - T_2 = -1 - 4 = -5$
$\therefore \quad a = 9, \quad \text{and} \quad d = -5$
The sequence is therefore an A.P. and hence,

$$\begin{aligned} T_{12} &= a + (12-1)d \\ &= 9 + 11(-5) \\ &= -46 \\ \text{Thus,} \qquad T_n &= a + (n-1)d \\ &= 9 + (n-1)(-5) \\ &= 14 - 5n \end{aligned}$$

(ii) Here, $T_1 = -x = a, T_2 = x + 3,$
$\therefore \quad T_2 - T_1 = (x+3) - (-x) = 2x + 3 = d;$
thus, $T_3 - T_2 = 2x + 3$
So, the sequence is an A.P., and therefore we have;

$$\begin{aligned} T_{12} &= a + (12-1)d \\ &= -x + (12-1)(2x+3) \\ &= -x + 11(2x+3) \\ &= 21x + 33 \\ &= 3(7x + 11) \end{aligned}$$

So that,

$$\begin{aligned} T_n = a + (n-1)d &= -x + (n-1)(2x+3) \\ &= 2nx - 3x + 3n - 3 \\ &= x(2n-3) + 3(n-1) \end{aligned}$$

Example 6.3.2

If the 7^{th} and 15^{th} terms of an $A.P.$ is 4 and 8 respectively, find the $A.P.$

Solution

$$\begin{aligned} T_7 &= a + (7-1)d \\ &= 4 \\ \text{i.e.} \quad a + 6d &= 4 \qquad\qquad (i) \end{aligned}$$

Similarly,

$$\begin{aligned} T_{15} &= a + (15-1)d \\ &= 8 \\ \text{i.e.} \quad a + 14d &= 8 \qquad\qquad (ii) \end{aligned}$$

Solving (i) and (ii) simultaneously for a and d we have;
$a = 1,\ d = \frac{1}{2}$.

$$\begin{aligned} \therefore \qquad T_n = a + (n-1)d &= 1 + (n-1)\frac{1}{2} \\ &= \frac{n+1}{2} \end{aligned}$$

Example 6.3.3

The sum of the three consecutive terms of an $A.P.$ is -6 and their product is 64, find the terms.

Solution

We could consider the terms as $a, a+d, a+2d$, but it is simpler to take them as $a-d, a, a+d$, i.e., by counting a step backwards or by subtracting d from the three terms, so that their sum becomes;

$$\begin{aligned} (a-d) + a + (a+d) &= -6 \\ \text{i.e.} \quad 3a &= -6 \quad \text{or} \quad a = -2 \end{aligned}$$

and their product as;

$$\begin{aligned} (a-d)\cdot a \cdot (a+d) &= 64 \\ \text{i.e.} \quad a(a^2 - d^2) &= 64 \end{aligned}$$

But we already have $a = -2,$ so that $\quad d = 6$

$$\therefore \qquad \text{the three terms are } -8, -2, 4$$

6.3.2 Arithmetic Mean (A.M.)

Suppose a, b, c are three consecutive terms of an $A.P.$, then the common difference is either $b - a$ or $c - b$ i.e.

$$\begin{aligned} d = b - a &= c - b \\ \text{or} \quad 2b &= a + c \end{aligned}$$

$$\therefore \qquad b = \frac{a + c}{2} \tag{6.3.2}$$

Thus, b is the required arithmetic mean of a and c. So, b is the average of a and c. More generally, the arithmetic mean between a and c are the n quantities that, when inserted between a and c, form with them $(n + 2)$ consecutive terms of an $A.P.$ Suppose d is the common difference of the $A.P.$ formed, then c is the $(n + 2)^{th}$ term of the $A.P.$ So that;

$$\begin{aligned} c = T_{n+2} &= a + [(n + 2) - 1]d \\ &= a + (n + 1)d \\ \text{i.e.} \quad d(n + 1) &= c - a \end{aligned}$$

$$\therefore \qquad d = \frac{c - a}{n + 1} \tag{6.3.3}$$

So that the required $A.M.$ are;

$$a + d, \quad a + 2d, \ldots, a + nd$$

where $T_1 = a$ and $T_{n+2} = c$ respectively.

Example 6.3.4

Insert 4 arithmetic means between,

(i) 10 and -10

(ii) x and $x + 15$.

Solution

(i) $a = 10, \quad T_6 = -10$
So that,

$$\begin{aligned} T_6 = a + (6-1)d &= -10 \\ \text{i.e.,} \quad 10 + 5d &= -10 \\ \therefore \quad d &= -4 \end{aligned}$$

Thus, the required numbers are,

$$a + (-4),\ a + 2(-4),\ a + 3(-4),\ a + 4(-4)$$

i.e. $6, 2, -2, -6$.

(ii) $a = x, \quad T_6 = x + 15$

$$\begin{aligned} \text{i.e.,} \quad T_6 = a + (6-1)d &= x + 5d \\ &= x + 15 \\ \therefore \quad d &= 3 \end{aligned}$$

$\therefore$ the required terms are $a + d,\ a + 2d,\ a + 3d,\ a + 4d$
Thus, on substituting for d we have the terms as;
$x + 3,\ x + 6,\ x + 9,\ x + 12$

6.3.3 Arithmetic Series

An arithmetic series is the sum of the terms of an arithmetic progression. An infinite arithmetic series can be written as

$$S_\infty = a + (a + d) + (a + 2d) + \ldots + [a + (n-1)d] + \ldots \qquad (6.3.4)$$

while a finite arithmetic series of n terms is written as

$$S_n = a + (a + d) + (a + 2d) + \ldots + [a + (n-1)d] \qquad (6.3.5)$$

Theorem 6.3.2. *To find formula for the sum of the first n terms of an A.P. with first term given as a and common difference as d.*

Proof. We already know that,

$$S_n = a + (a + d) + (a + 2d) + \ldots + [a + (n - 1)d]$$

If we write the right hand side (R.H.S.) of (6.3.5) in reverse order, we would have;

$$S_n = [a + (n - 1)d] + [a + (n - 2)d] + \ldots + (a + d) + a \qquad (6.3.6)$$

By adding (6.3.5) and (6.3.6);

$$\begin{aligned} 2S_n &= [2a + (n - 1)d] + (2a + (n - 1)d] + \ldots + [2a + (n - 1)d] \\ &\quad +[2a + (n - 1)d] \quad \text{to } n \text{ times} \\ &= n[2a + (n - 1)d] \end{aligned}$$

$$\therefore \quad S_n = \frac{n}{2}[2a + (n - 1)d] \qquad (6.3.7)$$

The result can be re-written as;

$$S_n = \frac{n(a + \ell)}{2} \qquad (6.3.8)$$

where $\ell = 2a + (n - 1)d$, i.e. the last term. The second result is more convenient when the first and the last terms are known. □

Example 6.3.5

Find the sum of the first 13 terms of;

(i) $1 + 2 + 3 + 4 + \ldots$

(ii) $(2x + 1) + 3x + (4x - 1) + \ldots$

Solution

(i) $a = 1, \quad d = 1;$
Thus,

$$\begin{aligned} S_{13} &= \frac{13}{2}[2a + (13 - 1)d] \\ &= \frac{13}{2}[2 + 12(1)] \\ \therefore \quad S_3 &= 91 \end{aligned}$$

(ii) $a = 2x + 1, \quad d = 3x - (2x + 1) = (4x - 1) - 3x = x - 1$
Thus,

$$\begin{aligned} S_{13} &= \frac{13}{2}[2a + (13 - 1)d] \\ &= \frac{13}{2}[2(2x + 1) + 12(x - 1)] \\ &= \frac{13}{2}(16x - 10) \\ &= 13(8x - 5) \\ &= 104x - 65 \end{aligned}$$

Example 6.3.6

The sum of n terms of an $A.P.$ is;

(i) $8n^2 - n$. Find the number of term which has the value 263.

(ii) $n(n + 1)$. Find the series in the $A.P.$

Solution

(i) $S_n = 8n^2 - n$.
Let r be an arbitrary term of the $A.P.$, then,

$$\begin{aligned} T_r = S_r - S_{r-1} &= 8r^2 - r - [8(r - 1)^2 - (r - 1)] \\ &= 16r - 9 \end{aligned}$$

So, suppose $T_r = 263$, we have;

$$\begin{aligned} 16r - 9 &= 263 \\ \therefore \quad r &= 17 \end{aligned}$$

$\therefore \quad 17^{th}$ term has the value 263.

(ii) Let r be an arbitrary term of the $A.P.$, thus;

$$\begin{aligned} T_r = S_r - S_{r-1} &= r(r + 1) - (r - 1)(r - 1 + 1) \\ &= 2r \end{aligned}$$

Hence,

$$\begin{aligned} T_1 &= 2(1) = 2 \\ T_2 &= 2(2) = 4 \\ T_3 &= 2(3) = 6 \end{aligned}$$

$\therefore$ the series is $2 + 4 + 6 + \dots$

Example 6.3.7

(i) The sum of the first 4 terms of an *A.P.* is 4, and the sum of the first 12 terms is 28. Find the A.P.

(ii) The sums of the first 5 and first 10 terms respectively of an *A.P.* are equal in magnitude but opposite in sign. Suppose the first term of the A.P. is 11, find the second term.

Solution

(i)

$$\begin{aligned} S_4 &= \frac{4}{2}[2a + (n-1)d] \\ &= 2(2a + 3d) \\ &= 4 \\ \text{i.e.} \quad 4a + 6d &= 4 \quad \text{or} \quad 2a + 3d = 2 \qquad (i) \end{aligned}$$

Similarly,

$$\begin{aligned} S_{12} &= \frac{12}{2}[2a + (12-1)d] \\ &= 6(2a + 11d) \\ &= 28 \\ \text{i.e.} \quad 6a + 33d &= 14 \qquad (ii) \end{aligned}$$

Solving (i) and (ii) simultaneously, we have;

$$a = \frac{1}{2}, \quad d = \frac{1}{3}$$

$\therefore$ the *A.P.* is $\frac{1}{2}, \frac{5}{6}, 1\frac{1}{6}, 1\frac{1}{2}, \dots$

(ii) $a = 11$, then,

$$\begin{aligned} S_5 &= \frac{5}{2}[2(11) + 4d] \\ &= \frac{5}{2}(22 + 4d) \\ &= 5(11 + 2d) \end{aligned}$$

Similarly,

$$\begin{aligned} S_{10} &= \frac{10}{2}[2(11) + 9d] \\ &= 5(22 + 9d) \end{aligned}$$

But since S_5 and S_{10} are equal in magnitude but different in sign, then we have;
$S_{10} = -S_5$,
So that,

$$\begin{aligned} 5(22 + 9d) &= -5(11 + 2d) \\ \text{i.e.} \quad 22 + 9d &= -11 - 2d \\ d &= -3 \end{aligned}$$

$$\begin{aligned} \therefore \quad T_2 &= a + d \\ &= 11 - 3 = 8 \end{aligned}$$

6.4 Geometric Progression (G.P.)

Geometric progression (G.P.) is the arrangement of numbers (terms) in a particular order in which each term is the product of a constant factor and the preceding term. We shall be discussing G.P. under two categories viz: Geometric Sequences and Geometric Series as we did in the previous section.

6.4.1 Geometric Sequences

A geometric sequence is a sequence of terms that increase or decrease by a constant ratio. The constant ratio is known as the *common ratio* of the sequence and is usually denoted by r. For example

(i) $16, -8, 4, -2, \ldots$

(ii) $3, 9, 27, 81, \ldots$

In (i), the common ratio is $-\frac{1}{2}$, while the common ratio in (ii) is 3.

Theorem 6.4.1. *To find a formula for the n^{th} term of a G.P., with common ratio given as r and first term as a.*

Proof. We know already that $T_1 = a$ i.e. first term. Consequently, we have;

$$\begin{aligned} T_1 &= a \\ T_2 &= T_1.r = ar \\ T_3 &= T_2.r = ar.r = ar^2 \\ &\vdots \\ T_m &= T_{m-1}.r = ar^{m-2}.r = ar^{m-2+1} = ar^{m-1} \\ &\vdots \\ T_n &= ar^{n-1} \end{aligned} \tag{6.4.1}$$

□

Example 6.4.1

Determine whether each of the sequence,

(i) $1, 2, 4, 8, \ldots$

(ii) $16, 8y, 4y^2, \ldots$

is *a* G.P. and hence find the 5th term for both cases. Determine the formula for the n^{th} term of (i) and (ii) respectively.

Solution

(i) This is a G.P., since,

$$\begin{aligned} \frac{T_2}{T_1} &= \frac{ar}{a} \\ &= r = 2 \end{aligned}$$

So, $T_1 = a = 1, \quad r = 2$

$$\begin{aligned} \therefore \qquad T_5 &= ar^{5-1} \\ &= (1)2^4 = 16 \end{aligned}$$

Hence,

$$\begin{aligned} T_n &= ar^{n-1} \\ &= (1)2^{n-1} \\ &= 2^{n-1} \end{aligned}$$

(ii)

$$\frac{8y}{16} = \frac{y}{2}$$

So, $a = 16, \quad r = \frac{y}{2}$, hence the sequence is a G.P.

$$\begin{aligned} \therefore \qquad T_5 &= ar^{5-1} \\ &= 16\left(\frac{y}{2}\right)^4 \\ &= y^4 \end{aligned}$$

and

$$\begin{aligned} T_n &= ar^{n-1} \\ &= 16\left(\frac{y}{2}\right)^{n-1} \\ &= \frac{2^4 y^{n-1}}{2^{n-1}} \\ &= \frac{y^{n-1}}{2^{n-5}} \end{aligned}$$

Example 6.4.2

If the 3rd and 6th terms of a G.P. is 8 and $\frac{64}{125}$ respectively, find the n^{th} term of the G.P.

Solution

$$\begin{aligned} T_3 &= ar^{3-1} \\ &= ar^2 = 8 \end{aligned}$$

Similarly,

$$\begin{aligned} T_6 &= ar^{6-1} \\ &= ar^5 = \frac{64}{125} \end{aligned}$$

Dividing, we have;

$$\begin{aligned} \frac{T_6}{T_3} &= \frac{ar^5}{ar^2} \\ &= \frac{\frac{64}{125}}{8} \\ \text{i.e.} \quad r^3 &= \frac{8}{125} \\ \therefore \quad r &= \pm\frac{2}{5} \end{aligned}$$

We can now solve for a as follows:

$$\begin{aligned} a\left(\pm\frac{2}{5}\right)^2 &= 8 \\ \text{i.e.,} \quad a &= 50 \end{aligned}$$

But since a is not negative, then $r = \frac{2}{5}$
It therefore follows that,

$$\begin{aligned} T_n &= ar^{n-1} \\ &= 50\left(\frac{2}{5}\right)^{n-1} \\ &= \frac{2(5^2)(2^{n-1})}{5^{n-1}} \\ &= \frac{2^{n-1+1}}{5^{n-1-2}} \\ &= \frac{2^n}{5^{n-3}} \end{aligned}$$

Example 6.4.3

The sum of the 1st and 3rd terms of a G.P. is 5 and the sum of the 2nd and 4th terms is 10. Find the formula for the n^{th} term of the G.P. and hence find the 1st 4 terms of the G.P.

Solution

If $T_1 = a$, and r is the ratio, then;

$$\begin{aligned} T_3 &= ar^2 \\ \text{and,} \quad a + ar^2 &= 5 \qquad (i) \\ \therefore \quad a &= \frac{5}{1+r^2} \end{aligned}$$

Similarly, $T_2 = ar$ and $T_4 = ar^3$, so that;

$$ar + ar^3 = 10 \qquad (ii)$$

Now, substituting for a in (ii) we have;

$$\begin{aligned} \frac{5r}{1+r^2} + \frac{5r^3}{1+r^2} &= 10 \\ \text{i.e.} \quad 5r + 5r^3 &= 10 + 10r^2 \\ r^3 - 2r^2 + r - 2 &= 0 \qquad (iii) \end{aligned}$$

Solving (iii) quadratically, we have; $r = 2$ or $\pm\sqrt{-1}$
We take $r = 2$ since the other values are complex numbers.
By solving for a in (i), we obtain;

$$\begin{aligned} a &= \frac{5}{1+4} \\ &= 1 \\ \therefore \quad T_n &= ar^{n-1} \\ &= (1) \cdot 2^{n-1} \\ &= 2^{n-1} \end{aligned}$$

So that the first 4 terms becomes $1, 2, 4, 8, \ldots$

Example 6.4.4

Find the value of x when $x+1,\ x+3$ and $x+7$ are the first 3 terms of a G.P. Obtain the n^{th} term of the G.P.

Solution

We are already given that, $T_1 = a = x + 1$ and $T_2 = ar = x + 3$. We can now solve for r as follows:

$$\begin{aligned} r &= \frac{T_2}{T_1} \\ &= \frac{ar}{a} \\ \therefore \qquad r &= \frac{x+3}{x+1} \end{aligned}$$

So that, from $T_3 = ar^2 = x + 7$, we have;

$$\begin{aligned} T_3 = (x+1)\left(\frac{x+3}{x+1}\right)^2 &= x+7 \\ \text{i.e.,} \quad (x+3)^2 &= (x+7)(x+1) \\ \therefore \qquad x &= 1 \end{aligned}$$

We now solve for r as;

$$\begin{aligned} r &= \frac{x+3}{x+1} \\ &= \frac{1+3}{1+1} \\ &= 2 \end{aligned}$$

and a as;

$$\begin{aligned} a &= x+1 \\ &= 2 \end{aligned}$$

So that,

$$\begin{aligned} T_n &= ar^{n-1} \\ &= 2(2^{n-1}) \\ &= 2^{n-1+1} \\ &= 2^n \end{aligned}$$

6.4.2 Geometric Mean

Suppose a, b, c are three successive terms of a G.P., then the common ratio is either b/a or c/b, such that,

$$\begin{aligned} \frac{b}{a} &= \frac{c}{b} \\ \text{or} \quad b^2 &= ac \end{aligned}$$

$$\therefore \qquad b = \pm\sqrt{ac} \tag{6.4.2}$$

Thus, b is the required geometric mean of a and c. More generally, the geometric means between a and c are the quantities that when inserted between a and c, form with them $(n+2)$th term of the G.P. Thus,

$$\begin{aligned} c &= T_{n+2} \\ &= ar^{n+2-1} \\ &= ar^{n+1} \\ \text{i.e.} \quad r^{n+1} &= \frac{c}{a} \end{aligned}$$

$$\therefore \qquad r = \sqrt[(n+1)]{\frac{c}{a}} \tag{6.4.3}$$

So that the required geometric means between a and c are;

$$ar,\ ar^2,\ ar^3, \ldots, ar^n$$

where $T_1 = a$ and $T_{n+2} = c$ respectively.

Example 6.4.5

(i) Insert 3 geometric means between $-\frac{1}{8}$ and -2.

(ii) If the geometric mean of $\frac{1}{4}$ and 4 is $y + 3$, find the possible values of y.

Solution

(i) If the means are ar, ar^2, ar^3, then we have the G.P. as;
$-\frac{1}{8}, ar, ar^2, ar^3, -2.$
So, given that $T_1 = a = -\frac{1}{8}$,
Then,

$$\begin{aligned} T_{3+2} = T_5 = ar^{5-1} &= -2 \\ \text{i.e.,} \left(-\frac{1}{8}\right) r^4 &= -2 \\ r^4 &= 16 \\ \therefore \quad r &= \pm 2 \end{aligned}$$

If $r = 2$, then the geometric means are;

$$-\frac{1}{4}, -\frac{1}{2}, -1$$

Similarly, when $r = -2$, we have;

$$\frac{1}{4}, -\frac{1}{2}, 1$$

(ii) Given that $y + 3$ is the geometric mean, then;

$$\begin{aligned} y + 3 &= \pm\sqrt{ac} \\ &= \pm\sqrt{\frac{1}{4} \cdot 4} \\ &= \pm 1 \\ \therefore \quad y &= \pm 1 - 3 \end{aligned}$$

which gives $y = -2$ or -4.

6.4.3 Geometric Series

A geometric series is the sum of the terms of a geometric progression. An infinite geometric series can be written as;

$$S_\infty = a + ar + ar^2 + \ldots + ar^{n-1} + \ldots = a \sum_{p=1}^{\infty} r^{p-1} \qquad (6.4.4)$$

Theorem 6.4.2. *To find the formula for the sum of the first n terms of a G.P. given that the first term is a and the common ratio is r.*

Proof. We have already known that,

$$S_n = a + ar + ar^2 + \ldots + ar^{n-1}$$

$$\text{i.e.} \quad rS_n = ar + ar^2 + \ldots + ar^{n-1} + ar^n \tag{6.4.6}$$

Subtracting (6.4.6) from (6.4.5) gives;

$$\begin{aligned} S_n - rS_n &= a - ar^n \\ &= a(1 - r^n) \\ S_n(1 - r) &= a(1 - r^n) \end{aligned}$$

$$\therefore \quad S_n = \frac{a(1 - r^n)}{1 - r} \tag{6.4.7}$$

when $|r| < 1$,
and

$$S_n = \frac{a(r^n - 1)}{r - 1} \tag{6.4.8}$$

when $|r| > 1$. The symbol $|r|$ is known as the modulus or absolute value of r. □

Example 6.4.6

Find the expression for the sum of the n terms of the series,

(i) $3 + \frac{3}{2} + \frac{3}{4} + \frac{3}{8} + \ldots$

(ii) $1 + x + x^2 + x^3 + \ldots$

Solution

(i) Given that $a = 3$,then,

$$\begin{aligned} r &= \frac{\frac{3}{2}}{3} \\ &= \frac{1}{2} \end{aligned}$$

Hence,

$$\begin{aligned} S_n &= \frac{a(1-r^n)}{1-r}, \quad \text{since } |\frac{1}{2}| < 1 \\ &= \frac{3(1-(\frac{1}{2})^n)}{1-\frac{1}{2}} \\ &= 6(1-(\frac{1}{2})^n) \\ &= 6(1-2^{-n}) \end{aligned}$$

(ii) $a = 1$, then,

$$\begin{aligned} r &= \frac{x}{1} \\ &= x \end{aligned}$$

Thus,

$$\begin{aligned} S_n &= \frac{a(1-r^n)}{1-r} \\ &= \frac{1-x^n}{1-x}, \quad \text{for } |x| < 1, \\ \text{or} \quad S_n &= \frac{x^n-1}{x-1}, \quad \text{for } |x| > 1 \end{aligned}$$

Example 6.4.7

The sum of the first 4 terms of a geometric series is 170 and the sum of the 1st and the 3rd term is 34. Show that there is one series satisfying these conditions, and find the sum of the first 6 terms of the series.

Solution

If we assume that $|r| > 1$, then,

$$\begin{aligned} S_4 &= \frac{a(r^4-1)}{r-1} \\ &= 170 \qquad (i) \end{aligned}$$

From $T_1 + T_3 = 34$, we have;

$$\begin{aligned} a + ar^2 &= 34 \\ a(1+r^2) &= 34 \\ \therefore \quad a &= \frac{34}{1+r^2} \end{aligned}$$

By substituting for a in (i), we have;

$$\begin{aligned} \frac{34}{1+r^2}\cdot\frac{r^4-1}{r-1} &= 170 \\ \frac{34(r^2-1)(r^2+1)}{(1+r^2)(r-1)} &= 170 \\ \frac{34(r-1)(r+1)(r^2+1)}{(1+r^2)(r-1)} &= 170 \\ 34(r+1) &= 170 \\ 34r+34 &= 170 \\ \therefore \quad r &= 4 \end{aligned}$$

Thus, a becomes;

$$\begin{aligned} a &= \frac{34}{1+4^2} \\ &= 2 \end{aligned}$$

Consequently,

$$\begin{aligned} T_n &= ar^{n-1} \\ &= 2(4^{n-1}) \\ &= 2^{2n-1} \end{aligned}$$

Hence the series is, $2 + 4 + 8 + 32 + \ldots$
So that,

$$\begin{aligned} S_6 &= \frac{2(4^6-1)}{4-1} \\ &= 2730 \end{aligned}$$

Note that if $r = 1$, then, the formula for S_n cannot be used. In this case however, the sum of the G.P. will become $S_n = na$.

Example 6.4.8

If the 6th term of a geometric series of positive numbers is 10 and the 16th term is 0.1, find the 11th term.

Solution

Given that,

$$\begin{aligned} T_6 = ar^5 &= 10 \\ \therefore \quad a &= \frac{10}{r^5} \end{aligned}$$

Similarly,

$$\begin{aligned} T_{16} = ar^{15} &= 0.1 \\ ar^{15} &= \frac{1}{10} \end{aligned}$$

Substituting for a in T_{16}, we have;

$$\begin{aligned} \frac{10}{r^5} \cdot r^{15} &= 10^{-1} \\ 10r^{10} &= 10^{-1} \\ r^{10} &= 10^{-2} \\ r &= 10^{-2(1/10)} \\ \therefore \quad r &= 10^{-1/5} \end{aligned}$$

On substitution for a, we have;

$$\begin{aligned} a &= \frac{10}{r^5} \\ &= \frac{10}{10^{-1/5(5)}} \\ &= \frac{10}{10^{-1}} \\ &= 10^2 \\ &= 100 \end{aligned}$$

Thus,

$$\begin{aligned} T_{11} &= ar^{10} \\ &= 10^2(10^{-1/5})^{10} \\ &= 10^2(10^{-2}) \\ &= 1 \end{aligned}$$

Example 6.4.9

The sum of the 1st 4 terms of a geometric series is 80. If the product of the 2nd and 4th terms of the G.P. doubles the 5th term, find the 1st 5 terms of the series and hence their sum.

Solution

Suppose we assume that $|r| > 1$, then;

$$S_4 = \frac{a(r^4 - 1)}{r - 1} = 80 \qquad (i)$$

If $T_2 \cdot T_4 = 2T_5$, then we have;

$$\begin{aligned} ar \cdot ar^3 &= 2ar^4 \\ a^2r^4 &= 2ar^4 \\ \therefore \quad a &= 2 \end{aligned}$$

By substituting for a in (i) we obtain;

$$\begin{aligned} \frac{2(r^4 - 1)}{r - 1} &= 80 \\ \frac{(r^2 - 1)(r^2 + 1)}{r - 1} &= 40 \\ \frac{(r - 1)(r + 1)(r^2 + 1)}{r - 1} &= 40 \\ \therefore \quad r^3 + r^2 + r + 1 &= 40 \qquad (ii) \end{aligned}$$

Solving equation (ii) we have;

$$r = 3 \text{ and } r^2 + 4r + 13 = 0$$

which implies that the other values of r is complex (please verify), hence for $r = 3$, we have the series as $2 + 6 + 18 + 54 + 162 + \dots$ Thus,

$$\begin{aligned} S_5 &= \frac{a(r^5 - 1)}{r - 1} \\ &= \frac{2(3^5 - 1)}{3 - 1} \\ &= 242 \end{aligned}$$

6.4.4 Infinite Geometric Series

Consider the formula for the sum of the 1st n terms of a geometric series;

$$\begin{aligned} S_n &= \frac{a(1 - r^n)}{1 - r}, \quad |r| < 1 \\ \text{and} \quad S_n &= \frac{a(r^n - 1)}{r - 1}, \quad |r| > 1 \end{aligned}$$

Obviously, the sum (S_n) of n terms is dependent on r^n in the formula above. When $|r| < 1$, as n increases r^n gets smaller and smaller (n being positive integer), and it can be said that r^n approaches zero as n approaches infinity i.e. $r^n \to 0$ as $n \to \infty$. Thus,

$$S_\infty = \frac{a(1 - r^n)}{1 - r} = \frac{a}{1 - r} - \frac{ar^n}{1 - r} \tag{6.4.9}$$

Observe that the second term becomes negligible as $r^n \to 0$.

Similarly when $|r| > 1$, as n increases r^n gets infinitely large, and it is then said that r^n approaches $\pm\infty$ as n approaches infinity i.e. $r^n \to \pm\infty$ as $n \to \infty$. Thus, S_∞ is then infinite i.e.

$$S_\infty = \pm\infty \tag{6.4.10}$$

However if $r = 1$, then every term in the series is the same as the first term, so that S_∞ is infinitely large. Similarly if $r = -1$, then the series becomes;

$$a - a + a - + \dots$$

Such that as n becomes infinitely large,

$$S_n = 0, \quad \text{if } n \text{ is even} \tag{6.4.11}$$

and

$$S_n = a \quad \text{if } n \text{ is odd} \tag{6.4.12}$$

The value that S_n approaches as n approaches infinity is known as it *sum to infinity* denoted by S_∞. If the sum to infinity is finite, as in the first case when $|r| < 1$, then the series is said to be *convergent* and we say that S_∞ converges to a limit. However when $|r| > 1$, the sum to infinity is said to be infinite i.e. $S_\infty = \pm\infty$ and the series is said to *diverge* i.e. no limit exists.

Example 6.4.10

If the 3rd term of a geometric series is 3/4, and the 6th term is $\frac{3}{32}$, find the 1st term, the common ratio and show, that however great n may be, the sum cannot exceed 6.

Solution

Given that,

$$\begin{aligned} T_3 = ar^2 &= \frac{3}{4} \\ \text{i.e.} \quad a &= \frac{3}{4r^2} \end{aligned}$$

Thus,

$$\begin{aligned} T_6 &= ar^5 \\ &= \frac{3}{32} \end{aligned}$$

Substituting for a in T_6, we have;

$$\begin{aligned} \frac{3}{4r^2} \cdot r^5 &= \frac{3}{32} \\ r^3 &= \frac{1}{8} \\ &= 2^{-3} \\ \therefore \quad r &= \frac{1}{2} \end{aligned}$$

So that;

$$\begin{aligned} a &= \frac{3}{4r^2} \\ &= \frac{3}{4 \cdot (\frac{1}{2})^2} \\ &= 3 \end{aligned}$$

Hence;

$$\begin{aligned} S_\infty &= \frac{a}{1-r}, \quad (\text{since } |r| < 1) \\ &= \frac{3}{1-\frac{1}{2}} \\ &= 6 \end{aligned}$$

$\therefore$ as $n \to \infty$, $S_\infty = 6$, which is the limit of S_n as $n \to \infty$.

Example 6.4.11

Write the repeating decimal $0.454545\ldots$, as a fraction.

Solution

Re-writing the decimal in series form we have;

$$0.454545\ldots = 0.45 + 0.0045 + 0.000045 + \ldots$$

So that,

$$\begin{aligned} r &= \frac{0.0045}{0.45} \\ &= 0.01 \end{aligned}$$

To write the repeating decimal as a fraction, we need to find the sum to infinity i.e. S_∞ of the series by substituting for a and r as follows;

$$\begin{aligned} S_\infty &= \frac{a}{1-r} \\ &= \frac{0.45}{1-0.01} \\ &= \frac{0.45}{0.99} \\ &= \frac{45}{99} \end{aligned}$$

6.5 Theory of Mathematical Induction

Logically, induction is a type of argument, and involves both a premise and a conclusion. It proceeds from a particular instance to a general conclusion. If the premise is true, then the conclusion as a matter of necessity must be absolutely true or certain. The premise must provide sufficient ground for accepting or affirming the conclusion. Those readers who are not too familiar with logical statements could go the Section 13.1 in Chapter 13 of this book for a better understanding of inductive argument.

Thus, the principle of mathematical induction is synonymous to inductive arguments logically, but as far as this topic is concerned, our purpose will principally be on numerical inductive arguments. Suppose that S is a set and let

(i) $I \in S$

(ii) $K \in S$ implies that $K+1 \in S$.

If we assume that the set S is valid for (i) and (ii), and suppose there exists a positive integer n such that $n \in S$. Now from (i) we know that $I \in S$. Further, since $I \in S$, we can infer from (ii) that $I+1 \in S$, and thus that $(I+1)+1 \in S$ and so on. Obviously, since n can be obtained from I by adding 1 successively $(n-1)$ times, then $n \in S$. We shall illustrate by examples as follows:

Example 6.5.1

Prove by induction that the sum of the 1st n natural numbers;

$$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$$

Solution

We need to first show that the result is valid for $n = 1$, i.e.

$$\frac{1(1+1)}{2} = 1$$

So it is true for $n = 1$.
Next, for $n = 1 + 1$, we have;

$$1 + 2 = \frac{2(2+1)}{2} = 3$$

$\therefore$ it is equally true for $n = 1, 2$,
We now assume that it is also true for any arbitrary value, say $n = N$, such that,

$$1 + 2 + 3 + \ldots + N = \frac{N(N+1)}{2}$$

Next, put $n = N + 1$ and add the $N + 1$ to both sides:

$$\begin{aligned}
1 + 2 + 3 + \ldots + N + (N+1) &= \frac{N(N+1)}{2} + (N+1) \\
&= \frac{N(N+1) + 2(N+1)}{2} \\
&= \frac{(N+1)(N+2)}{2} \\
&= \frac{1}{2}(N+1)[(N+1)+1]
\end{aligned}$$

$\therefore$ the result is valid for $n = N + 1$.
It therefore follows from the foregoing that by the principle of mathematical induction, the result is true for all values of n

Example 6.5.2

Prove by induction that, if n be a positive integer, the sum,

$$1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{1}{6}n(n+1)(2n+1)$$

Solution

For $n = 1$, we have;

$$\begin{aligned} 1^2 &= \frac{1}{6}(1)(1+1)(2(1)+1) \\ &= \frac{1}{6}(2)(3) \\ &= 1 \end{aligned}$$

$\therefore$ the result is true for $n = 1$
Next, for $n = 1 + 1 = 2$,

$$\begin{aligned} 1^2 + 2^2 &= \frac{1}{6}(2)(2+1)(2(2)+1) \\ &= \frac{1}{6}(2)(3)(5) \\ &= 5 \end{aligned}$$

$\therefore$ it is also true for $n = 2$.
Now, suppose p is an arbitrary value in the set, such that;

$$1^2 + 2^2 + 3^2 + \ldots + p^2 = \frac{1}{6}p(p+1)(2p+1)$$

It follows that,

$$\begin{aligned}
1^2 + 2^2 + \cdots + p^2 + (p+1)^2 &= \frac{1}{6}p(p+1)(2p+1) + (p+1)^2 \\
&= \frac{p(p+1)(2p+1) + 6(p+1)^2}{6} \\
&= \frac{(p+1)[p(2p+1) + 6(p+1)]}{6} \\
&= \frac{(p+1)(2p^2 + p + 6p + 6)}{6} \\
&= \frac{(p+1)(2p^2 + 7p + 6)}{6} \\
&= \frac{(p+1)(p+2)(2p+3)}{6} \\
&= \frac{1}{6}(p+1)[(p+1)+1][2(p+1)+1]
\end{aligned}$$

$\therefore$ the result is true for $n = p + 1$. Hence it is equally true for all values of n.

Example 6.5.3

Show that if n is a positive integer, then the sum of the cubes of the 1st n terms is $\frac{1}{4}n^2(n+1)^2$.

Solution

Let $\mathbb{Z}$ be the set of the positive integers, then;

$$1^3 + 2^3 + \ldots + n^3 = \frac{1}{4}n^2(n+1)^2$$

Now for $n = 1$,

$$\begin{aligned}
1^3 &= \frac{1}{4}(1)^2(1+1)^2 \\
&= \frac{4}{4} \\
&= 1
\end{aligned}$$

$\therefore$ $n = 1$ is true for the result.
Next for $n = 2$,

$$\begin{aligned} 1^3 + 2^3 &= \frac{1}{4}(2)^2 \cdot (2+1)^2 \\ &= \frac{1}{4}(4)(9) \\ &= 9 \end{aligned}$$

$\therefore$ it is also true for $n = 2$.
Now, assume k to be any arbitrary integer such that $K \in \mathbb{Z}$, then;

$$1^3 + 2^3 + 3^3 + ... + k^3 = \frac{1}{4}k^2(k+1)^2$$

So, for $n = k + 1$, add $(k + 1)^3$ to both sides to obtain,

$$\begin{aligned} 1^3 + 2^3 + 3^3 + \ldots + k^3 + (k+1)^3 &= \frac{1}{4}k^2(k+1)^2 + (k+1)^3 \\ &= \frac{1}{4}[k^2(k+1)^2 + 4(k+1)^3] \\ &= \frac{1}{4}(k+1)^2(k^2 + 4k + 4) \\ &= \frac{1}{4}(k+1)^2(k+2)(k+2) \\ &= \frac{1}{4}(k+1)^2(k+2)^2 \\ &= \frac{1}{4}(k+1)^2[(k+1)+1]^2 \end{aligned}$$

$\therefore$ the statement is true for $n = k+1$, and it is equally true for $n = k$. Thus by the principle of mathematical induction, it is also true for all values of n.

Chapter 7

Combinatorics

Combinatorics is an aspect of mathematics that deals with the act or process of combining items or elements in a group/set. Elements in a group could be combined by arranging according to a particular order i.e. ascending or descending, or without order i.e. random. This topic has a wide range of applications in our everyday life. For instance, students sitting for an examination can be arranged in increasing number of their matriculation numbers in an examination hall. Similarly, it is applied in the industries for production planning and in social activities for event management. For example, in a Job shop operation system, jobs are assigned to machines according to a particular order such as First Come First Serve (FCFS), Shortest Processing Time (SPT) etc.

7.1 Factorial Notation

We define factorial n symbolized by $n!$, as the product of the positive integers from 1 to n, inclusive and is of the form

$$n! = 1.2.3 \ldots (n-2)(n-1)n \tag{7.1.1}$$

Theorem 7.1.1. *To prove that* $n! = 1.2.3 \ldots (n-2)(n-1)n$

Proof. The proof of this theorem is by *mathematical induction.* We need to show that if the statement holds true for $n = k$, it must also be true for $n = k+1$. If we can prove this, we shall have achieved our objective.

Having arbitrarily arranged the objects $1, \dots, k$, we still have to place the object $k+1$. This can be done in $k+1$ different ways, namely: before the first place, before the second place, ... before the $k^t h$ place, and after the k^{th} place. If the number of ways of arranging k objects was;

$$1.2.3.\dots k,$$

then for $k+1$ objects, it is $(k+1)$ times as large, i.e.;

$$1.2.3.\dots k.(k+1)$$

This completes the proof. □

Corollary 7.1.1.

$$\begin{aligned} n! &= n(n-1)! \\ &= n(n-1)(n-2)! \\ &= n(n-1)(n-2)(n-3)! \\ &= n(n-1)(n-2)(n-3)\dots(n-r+1)! \end{aligned} \tag{7.1.2}$$

where r is any positive integer and $n \geq r$.

It follows that from the corollary 7.1.2 that;

$$\begin{aligned} 4! &= 4.3! \\ 3! &= 3.2! \\ 2! &= 2.1 \end{aligned}$$

If we carry this process forward in a mechanical manner, we would write,

$$1! = 1.0!$$

But in order for the symbol 0! to have meaning, we would let

$$0! = \frac{1!}{1} = 1 \tag{7.1.3}$$

We shall therefore define the value of this symbol 0! as equal to 1.

Example 7.1.1

(i)

$$\begin{aligned} 8! &= 8(8-1)(8-2)(8-3)(8-4)(8-5)(8-6)(8-7) \\ &= 8.7.6.5.4.3.2.1 \\ &= 40320 \end{aligned}$$

(ii)

$$\begin{aligned} 5! &= 5(5-1)(5-2)(5-3)(5-4) \\ &= 5.4.3.2.1 \\ &= 120 \end{aligned}$$

Corollary 7.1.2. *If n is a negative integer or rational, the factorial notation is of the form;*

$$\begin{aligned} (-n)! &= (-n)(-n-1)! \\ &= (-n)(-n-1)!(-n-2)! \\ &= (-n)(-n-1)!(-n-2)\ldots(-n-r+1)! \end{aligned} \tag{7.1.4}$$

Equivalently, if n is replaced by a rational, say $n = \frac{x}{y}$, then it follows from equation 7.1.4.

Theorem 7.1.4 (A Fundamental Assumption). *If an operation can be carried out in m different ways; and if after it is completed in one of these ways, a second operation can be carried out in n different ways; then the two operations together can be carried out in mn different ways in that order.*

Proof. If the first operation can be performed in any one way, then the second operation can be carried out in n ways; and thus since there are m ways of performing the first operation, each of these ways is associated with n ways of performing a second operation. Thus, there will be mn ways of performing the two operations simultaneously. □

Example 7.1.2

Virgin Nigeria has four airlines that flies between Lagos and Abuja. If a passenger travels from Lagos to Abuja by one of the airline and returns by another, in how many ways can he make the return journey.

Solution

The passenger can travel on any one of the four flights to Abuja, and has thus four ways of choosing his flight to Abuja. On his return journey, he has the choice of $(4 - 1)$ airliners, and each of these three ways can be associated with any one of the four ways of traveling from Lagos. Hence there are;

$$4 \times 3 = 12$$

ways of making the return journey.

7.1.1 Stirling Formula

Calculating $n!$ for large values of n is rather cumbersome. There is however, an approximation formula named after STIRLING:

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \tag{7.1.5}$$

where e is the base of the natural logarithms: $e = 2.71828...$ The symbol $\sim$ means that the quotient of the left and right members tends to 1 as $n \to \infty$; i.e. as n grows large, the percentage error in this approximation approaches zero:

$$\lim_{n\to\infty} \frac{n!}{\sqrt{2\pi}e^{-n}n^{n+1/2}} = 1 \tag{7.1.6}$$

Proof[1] **(Appendix)**

Remark 12 *From $n = 9$ onwards, STIRLING formula gives the value of $n!$ with an accuracy of better than 1%.*

Example 7.1.3

Using Stirling formula calculate 15! and 9!

Solution

(i)

$$\begin{aligned} 15! &\simeq \sqrt{2\pi.15}\left(\frac{15}{e}\right)^{15} \\ &\simeq \sqrt{30\pi}\left(\frac{15}{e}\right)^{15} \\ &\simeq 1.300\ 407 \times 10^{12} \end{aligned}$$

(ii) Actual, $9! = 362\ 880$
and the approximated 9! using Stirling's formula is;

$$9! \simeq \sqrt{18\pi}\left(\frac{9}{e}\right)^{9} \simeq 359\ 536.87$$

Differences = 3 343.13; which gives less than 1% percentage error.

7.2 Permutations (Arrangements)

Permutation is change in the order of sequence of elements or objects in a series; especially, the making of all possible changes of sequence. In other words, each arrangement that can be made by taking some or all of a number of different objects is known as *permutation*. We can simply refer to 'permutation' as 'arrangement'. Consider for example, the set of three letters a,b,c. The following are the arrangements of these letters taking all or some at a time.

(i) abc, acb, cab, cba, are permutations of the three letters (taken two at a time).

(ii) ab, ba, ac, ca, bc, cb, are permutations of the three letters (taken two at a time).

(iii) a, b, c, are the permutations of the three letters (taken one at a time).

The permutations of n objects taken r at a time can be denoted by

$$P(n, r) \text{ or } nPr \text{ or } Pn, r \text{ or } P_r^n \text{ or } (n)r.$$

However for uniformity, we shall restrict ourselves to the use of $P(n, r)$ remark throughout our discussion in this section.

Remark 13 *When considering the groups without the order of the letters being taken into account, the first three in (i) and (ii) above constitute the number of groups or (selections) of the three letters taken all or two at a time respectively.*

7.2.1 Fundamental Theorem on Permutations

Theorem 7.2.1. *The number of permutations possible of r objects taken from n different objects, (for n and r being positive integers with $n \geq r$) is given by;*

$$P(n, r) = \frac{n!}{(n-r)!}$$

Proof. To find an expression for $P(n, r)$, we observe that the first element in an r-permutation of n objects can be chosen in n different ways. Following this, the second element in the permutation can be chosen in $(n-1)$ ways; and thus the number of ways in the permutation of choosing the first two elements is $n(n-1)$.Furthermore, the third element in the permutation can be chosen in $(n-2)$ ways, each of which can be associated with the numbers of ways in the permutation of choosing the first two elements. Hence, by fundamental assumption, we may choose the first three objects or elements in $n(n-1)(n-2)$ ways.

Proceeding in this manner, we have that the rth (last) element in the r-permutation can be chosen in $[n-(r-1)]$ ie. $(n-r+1)$ ways and thus r elements may be selected in;

$$n(n-1)(n-2)\ldots(n-r+1)$$

ways.
Since the next factor in the order of this product would be $(n-r)$, we multiply this product by $\frac{(n-r)!}{(n-r)!}$ to obtain;

$$\frac{n(n-1)(n-2)\ldots(n-r+1)(n-r)!}{(n-r)!}$$

But the numerator is equivalent to $n!$, hence;

$$P(n,r) = \frac{n!}{(n-r)!} \tag{7.2.1}$$

□

Corollary 7.2.1

The number of permutations possible of n different objects taken all at a time is given as;

$$P(n,n) = n! \tag{7.2.2}$$

The proof is established when we set $r = n$ in $(n-r)!$ to obtain $0!$ or 1 as the denominator in (7.2.1) above.

Examples 7.2.1

(i) If five sprinters compete in a final race of 100 meters dash, in how may different ways may three prizes be won?

Solution

Here the order in which they finish determines the first, second, and third prize awards. Given that, $n = 5, r = 3$, hence by (7.2.1);

$$\begin{aligned} P(5,3) = \frac{5!}{2!} &= \frac{5(5-1)(5-2)(5-3)!}{2!} \\ &= 5.4.3 \\ &= 60 \end{aligned}$$

$\therefore$ The prizes may be won in 60 different ways.

(ii) How many different arrangements can be made from the letters 'STUDENT' by taking (a) four at a time? (b) all at a time?

Solution

(a) The numbers of letters in the word student, $n = 7$; and when we take four letters at a time, then $r = 4$, such that;

$$\begin{aligned} P(7.4) = \frac{7!}{(7-4)!} &= 7(7-1)(7-2)(7-3) \\ &= 7.6.5.4 \\ &= 840 \end{aligned}$$

(b) Taken all the letters at a time, then $n = 7$ and $r = 7$, we have;

$$\begin{aligned} P(7,7) &= 7! \\ &= 5\ 040 \text{ different arrangements} \end{aligned}$$

Theorem 7.2.2. *To prove that,* $P(n, r) = (n-r+1) \cdot P(n, r-1)$

Proof.

$$\begin{aligned} (n-r+1) \cdot P(n, r-1) &= (n-r+1) \cdot \tfrac{n!}{[n-(r-1)]!} \\ &= (n-r+1) \cdot \tfrac{n!}{(n-r+1)!} \\ &= (n-r+1) \cdot \tfrac{n(n-1)(n-2)...(n-r+1)(n-r)!}{(n-r+1)(n-r)!} \\ &= \tfrac{n(n-1)(n-2)...(n-r+1)(n-r)!}{(n-r)!} \end{aligned}$$

But recall that the numerator is equivalent to $n!$, it follows that;

$$(n-r+1) \cdot P(n, r-1) = \frac{n!}{(n-r)!} = P(n, r) \qquad (7.2.3)$$

which completes the proof. □

Theorem 7.2.3. *The number of permutations possible of* n *elements consisting of* k *different groups such that* n_1 *elements are* *alike in the first group, n_2 are alike in the second group, ..., and n_k are alike in the k^{th} group; where*

$$n_1 + n_2 + \ldots + n_k = N$$

is given by the formula;

$$P(N : n_1 n_2 \ldots n_k) = \frac{N!}{n_1! n_2!} \ldots n_k!$$

Proof. If the symbol on the RHS represents the desired number of permutations, and if each n_i elements $(i = 1, 2, \ldots, k)$, were replaced by n_i elements, all different from one another and from all other elements in the group of N. Then the number of permutations of N different elements taken all at a time could be obtained by multiplying $P(N : n_1, n_2 \ldots, n_k)$ by $n_1! n_2 \ldots n_k!$, for $n_i!$ represents the number of permutations possible from n_i different elements taken all at a time. Thus,

$$P(N_i : n_1, n_2, \ldots, n_k) n_1! n_2! \ldots n_k! = N!$$

whence formula (7.2.4) follows;

$$P(N : n_1, n_2, \ldots, n_k) = \frac{N!}{n_1! n_2! \ldots n_k!} \tag{7.2.4}$$

□

Example 7.2.2

(i) There are twelve students in a class. In how many ways can they take three different tests if four students are to take each test?

(ii) When a 10 kobo coin was tossed eight times in succession, head appeared three times and tail five times in the following order $THHTHTTT$. In how many other orders could they have appeared?

Solution

(i) Given that $N = 12$, if the 12 students are grouped into 3, with each group containing 4 students, then $n_1 = 4, n_2 = 4, n_3 = 4$, where $n_1 + n_2 + n_3 = N = 12$

$$\therefore \qquad P(12; 4, 4, 4) = \frac{12!}{4!4!4!} = 34\ 650 \text{ ways}$$

(ii) Here, there are $8! = 40\ 320$ permutations of the eight letters, where all the heads and tails are distinguished.

But where the heads and the tails are not distinguished, then the number of orders they would appear is given by;

$$\begin{aligned} \frac{8!}{3!5!} &= \frac{40\ 320}{6 \times 120} \\ &= 56 \end{aligned}$$

$\therefore$ the other orders that could have appeared is $56 - 1 = 55$ orders

Corollary 7.2.2. *The number of unordered divisions of n distinct elements into i_1 groups containing 1 element each, i_2 groups containing 2 elements each, ... i_n groups containing n elements each is;*

$$\frac{n!}{[(1!)^{i_1} i_1!][(2!)^{i_2} i_2!] \dots [(n!)^{i_n} i_n!]} \tag{7.2.5}$$

where $i_1, i_2, \dots, i_n$ are non-negative integers such that

$$1i_1 + 2i_2 + \dots n i_n = n \tag{7.2.6}$$

In counting the unordered divisions of n elements, it is convenient to characterize a division not by group sizes but what amounts to the same element - by the number i_1 of groups containing 1 element, the number i_2 of groups containing 2 elements, ..., and the number i_n of group containing n elements.

Example 7.2.3

(i) In how many ways can twelve oranges be divided among three children such that each child gets four oranges.

(ii) In how many ways can ten labourers be divided into four groups to do a job, such that two groups contain two men each and the other two groups contains three women each.

Solution

(i) For 3 groups of 4 oranges each, we have;

$$\frac{12!}{(4!)^3 3!} = 5\ 775 \text{ ways}$$

(ii) There are,

$$\frac{10!}{[(2!)^2 2!][(3!)^2 2!]} = 6\ 300 \text{ ways}$$

Theorem 7.2.4. *Let n be any arrangement of things with k distinct objects each with an infinite repetition number. Then the number of r-permutation of n is*

$$K^r$$

Proof. In constructing an r-permutation of n elements, we can choose the first item in k distinct ways, the second item in k distinct ways, ..., and the rth item in k distinct ways. Since all repetition numbers of n are infinite, the number of different choices for any item is always k and does not depend on the choices of the previous items. By fundamental assumption, the r items can be chosen in k^r ways. □

Example 7.2.4

How many numbers of two digits can be formed from the numbers $1, 2, 3, 4$, when each of these can be repeated up to four times? How many of these have two or less equal digits?

Solution

From the previous Theorem 7.2.6.,

$$4^2 = 16 \text{ numbers}$$

Thus, there will be $P(4,2)$ of these numbers having all different digits

$$P(4,2) = \frac{4!}{(4-2)!} = \frac{4!}{2!} = 12$$

$\therefore$ $16 - 12 = 4$ numbers will have two or less equal digits.

7.2.2 Permutations in a Ring (Cyclic Permutations)

The permutations that we have just considered are more properly called *linear permutations*. We think of the objects as being arranged in a line. If instead of arranging objects in a line, we arrange them in a circle, the number of permutations decreases. Let us consider it this way. Suppose eight (8) children are marching in a circle, in how many different ways can they form their circle?

Since the children are moving, what matters is their positions relative to each other and not to their environment. Thus we must regard two circular permutations as equal if one can be brought to the other by a rotation. Since each linear permutation of the children gives rise to seven (7) others by rotation, to find the number of circular permutations we must divide the number of linear permutation by 8. Thus the number of circular permutations of the 8 children equals,

$$\frac{8!}{8} = 7!$$

Theorem 7.2.5. *The number of circular r-permutations of a set of n objects is given by;*

$$\frac{P(n,r)}{r} = \frac{n!}{r(n-r)!}$$

In particular, the number of circular permutations of n objects is $(n-1)!$

Proof. A proof is essentially contained in the preceding section (7.2.6). The linear r-permutations can be grouped into sets of r such that each linear r-permutation in the same set gives rise to the same circular r-permutation. Since there are $P(n, r)$ linear r-permutations, the number of r-permutations is

$$\frac{P(n,r)}{r} = \frac{n!}{r(n-r)!} \tag{7.2.7}$$

□

Remark 14 *We will continue to use 'permutation' for 'linear permutation' as we did before in our discussion of circular permutations.*

Example 7.2.5

In the Directors Board meeting of Lee Engineering, there are ten members present; (a) in how many ways can they take positions at a round table? (b) how many ways are there if the Chairman and the Vice must sit next to each other? (c) how many ways are there if they (Chairman and the Vice) must not sit next to each other?

Solution

(a) Let one of them take a place (it does not matter where, as the positions of the others will be relative). Then there are 9 choices for the second person, 8 for the third person, ..., and 1 for the tenth persons.
$\therefore$ by our previous Theorem 7.2.5, the number of ways is given by;

$$\frac{P(10,10)}{10} = \frac{10!}{10} = 9! = 362880$$

(b) Let the Chairman and the Vice sit at the table next to each other. There are two ways for this, the vice Chairman being on the left or right of the Chairman. Then, the third member has a choice of (10-2) places and so on.

Hence the number of ways will be;

$$\begin{aligned} 2 \times (10-2)! &= 2 \times 8! \\ &= 80\ 640 \end{aligned}$$

(c) Using the solutions to (a) and (b) above, the number of ways in which the Chairman and the Vice must not sit next to each other will be;

$$362\ 880 - 80\ 640 = 282\ 240$$

Corollary 7.2.3. *If the n items in the circular r-permutations are fixed on a ring, which can be turned over, the number of r-permutations will be;*

$$\frac{(n-1)!}{2}$$

as an r-permutation with the ring facing one way will be one of the r-permutations or arrangements with the ring turned over.

Example 7.2.6

What is the number of necklaces that can be made from 16 beads each of different colour?

Solution

A necklace consists of 16 beads arranged in a circle. There are,

$$\frac{16!}{16} = \frac{15!}{2} = 6.5021535 \times 10^{11}$$

(Refer to STIRLING formula in 7.1.5)

Example 7.2.7

Six different keys are kept on a circular key holder. How many different arrangements of the keys are Possible?

Solution

There are $(6-1)! = 5!$ of such arrangements, since a key holder can be turned over without any change, then the numbers of arrangements are;

$$\frac{(6-1)!}{2} = \frac{5!}{2} = \frac{120}{2}$$
$$= 60$$

7.3 Combinations

The formulas developed in the previous section (7.2) serve very useful purposes for counting the number of ways that certain kinds of event may occur. However, there are other kinds of problems for which the formulas are not suited. Therefore, this section shall focus on the discussion of these kind of problems together with the methods for solving them.

Suppose we have a collection of n objects. A combination of these n objects taken r at a time is any selection of r of the objects where order does not count. Hence each of the groups or selections that can be made by taking some or all of a number of different objects is called *combination*. In other words, an r-combination of a set of n objects is any subset containing r object. For example, the combinations of the letters a, b, c, d taken three at a time are abc, abd, acd, bcd.
Observe that the following combinations are equal;

$$abc, acb, bac, bca, cab, cba.$$

That is, each set contains the same elements $\{a, b, c\}$. Symbolically, the number of n objects taken r at a time can be denoted by;

$$C(n, r),\ \binom{n}{r},\ {}^{n}C_{r},\ \text{or } C_{n,r}$$

We shall however, restrict our usage to $C(n, r)$ remarks throughout this section.

Remark 15 *As an aid to distinguishing between permutation and combinations, note that for permutations (arrangements), the order of the elements is of importance, while for combinations (selections) the order of the elements is not important.*

Theorem 7.3.1. *Let r be a non-negative integer, for $r \leq n$, the number of combinations of n different objects taken r at a time is given by*

$$C(n,r) = \frac{n!}{r!(n-r)!}$$

Proof. Let the number of combinations be y. Then each combination will give rise to $P(r,r)$ permutations, i.e. $r!$ permutations. Such that;

$$\begin{aligned} y.r! &= P(n,r) \\ &= n(n-1)(n-2)\ldots(n-r+1) \\ y &= C(n,r) \\ &= \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!} \\ &= \frac{n(n-1)(n-2)\ldots(n-r+1)(n-r)(n-r-1)\ldots2.1}{r!(n-r)(n-r-1)\ldots2.1} \\ &= \frac{n!}{r!(n-r)!} \end{aligned} \tag{7.3.1}$$

It follows that the number of combinations possible, using the elements from a set with n elements, such that only r objects are used in each combination, is the number of permutations of n elements taking r at a time divided by the number of permutations of a subset of r objects taking all r in each permutation. Symbolically, we can write the number of combinations, $C(n,r)$ as;

$$C(n,r) = \frac{P(n,r)}{P(r,r)} = \frac{P(n,r)}{r!} = \frac{n!}{r!(n-r)!}$$

□

Corollary 7.3.1. *The number of ways of choosing n objects from n different items is given by;*

$$C(n, n) = 1 \tag{7.3.2}$$

But,

$$C(n, r) = \frac{n!}{r!(n-1)!}$$

$$\therefore \qquad C(n, n) = \frac{n!}{n!(n-n)!} = \frac{1}{0!} = 1 \tag{7.3.3}$$

Theorem 7.3.2. *For all non-zero counting numbers n and r, $(0 < r < n)$, it is true that;*

$$C(n, r) = C(n, n-r)$$

Proof. One way to understand this relationship is to recall that each subset has a complementary set in a universal set. To verify this claim by a computation technique, simply observe that,

$$\begin{aligned} C(n, r) &= \frac{n!}{r!(n-r)!} \\ &= \frac{n!}{(n-r)!r!} \\ &= \frac{n!}{(n-r)![n-(n-r)]!} \\ &= C(n, n-r) \end{aligned}$$

□

Example 7.3.1

A student is to answer four questions out of six on a test. (a) How many choices has he? (b) How many if he must answer the last two questions? (c) How many if he must answer at least two in the first three questions?

Solution

(a) The four questions can be selected in,

$$\begin{aligned} C(n,r) &= C(6,4) \\ &= \frac{6!}{4!(6-4)!} \\ &= \frac{6!}{4!2!} \\ &= 15 \text{ ways} \end{aligned}$$

(b) He can choose the other two questions from the remaining four in,

$$C(4,2) = \frac{4!}{2!(4-2)!} = \frac{4!}{2!2!} = 6 \text{ ways}$$

(c) If he answers all the first three questions then he would be left with one out of the remaining three questions in,

$$C(3,3) = 1$$

and

$$C(3,1) = \frac{3!}{1!(3-1!} = \frac{3!}{1!2!} = 3 \text{ ways}$$

However if on the alternative, he decides to answer two in the first three questions, then he would be left with two out of the remaining three in,

$$C(3,2) = \frac{3!}{2!(3-2)!} = \frac{3!}{2!1!} = 3 \text{ ways}$$

and

$$C(3,2) = \frac{3!}{2!(3-2)!} = 3 \text{ ways}$$

Hence, he can choose the four questions in either,

$$(1 \times 3) \text{ ways, or } (3 \times 3) \text{ ways}$$

Which gives, $(3 + 9) = 12$ ways.

Example 7.3.2

The Personnel Manager of Coca-Cola has six applicants for two marketing jobs, how many different pairs of employees can he hire? How many different sets of four persons can he hire?

Solution

He can hire,

$$C(6,2) = \frac{6!}{2!(4-2)!} = \frac{6.5}{2!} = 15 \text{ pairs of employees}$$

However, if he needs a subset containing 4 persons, then he can hire,

$$C(6,4) = \frac{6!}{4!(6-4)!} = \frac{6.5}{2!} = 15 \text{ pairs of four persons}$$

Example 7.3.3

If examination disciplinary committee of the University of Abuja comprising of 4 members is to be formed from five Academic Staff, two Non-academic Staff and the Deputy Vice-Chancellor (Academic). In how many ways can the committee be formed if (a) it must include the deputy Vice-Chancellor; (b) the deputy vice-chancellor is not to be included; (c) exactly two academic staff must be included; (d) not more than three academic staff are to be included?

Solution

(a) The options available are either,

(i) All the 3 remaining members are academic staff i.e.,

$$C(5,3) = \frac{5!}{3!(5-3)!} = \frac{5!}{3!2!} = \frac{5!}{3!2!} = 10 \text{ ways}$$

or

(ii) Two academic staff and one non-academic staff i.e.,

$$\begin{aligned} C(5,2) \times C(2,1) &= \frac{5!}{2!(5-2)!} \times \frac{2!}{1!(2-1)!} \\ &= 10 \times 2 \\ &= 20 \text{ ways} \end{aligned}$$

or

(iii) One academic staff and two non-academic staff i.e.;

$$C(5,1) \times C(2,2) = 5 \times 1 = 5 \text{ ways}$$

$\therefore$ the committee can be formed in,

$$10 + 20 + 5 = 35 \text{ ways}$$

(b) The committee may comprise of,

(i) Four academic staff only i.e.,

$$C(5,4) = \frac{5!}{4!(5-4)!} = \frac{5!}{4!1!} = 5 \text{ ways}$$

(ii) Three academic and one non-academic i.e.,

$$C(5,3) \times C(2,1) = \frac{5!}{3!(5-3)!} \times 2 = \frac{5!}{3!2!} \times 2 = 20 \text{ ways}$$

(iii) Two academic and two non-academics i.e.,

$$C(5,2) \times C(2,2) = \frac{5!}{2!(5-3)!} \times 1 = 10 \text{ ways}$$

$\therefore$ the committee can be formed in:
$5 + 20 + 10 = 35$ ways.

(c) The members may comprise of;

(i) Two academic and two non-academic i.e.,

$$C(5,2) \times C(2,2) = \frac{5!}{2!(5-2)!} \times 1 = \frac{5!}{2!3!} = 10 \text{ ways}$$

or

(ii) Two academic, one non-academic and the Deputy Vice-Chancellor i.e.,

$$\begin{aligned} C(5,2) \times C(2,1) \times C(1,1) &= \frac{5!}{2!(5-2)!} \times 2 \times 1 \\ &= 20 \text{ ways} \end{aligned}$$

$\therefore$ the committee can be formed in $10+20 = 30$ ways.

(d) The members may include either;

(i) One academic, two non-academics and the DVC i.e.,

$$C(5,1) \times C(2,2) \times C(1,1) = 5 \times 1 \times 1 = 5 \text{ ways .}$$

or

(ii) Two academic and two non-academics i.e.,

$$C(5,2) \times C(2,2) = \frac{5!}{2!(5-2)!} \times 1 = 10 \text{ ways}$$

or

(iii) Two academics, one non-academic and the DVC i.e.,

$$C(5,2) \times C(2,1) \times C(1,1) = 10 \times 2 \times 1 = 20 \text{ ways}$$

or

(iv) Three academics and the DVC i.e.,

$$C(5,3) \times C(1,1) = 10 \times 1 = 10 \text{ ways}$$

or

(v) Three academic and one non-academic staff i.e.,

$$C(5,3) \times C(2,1) = 10 \times 2 = 20 \text{ ways}$$

$\therefore$ the committee can be formed in,

$$5 + 10 + 20 + 10 + 20 = 65 \text{ ways}$$

Theorem 7.3.3. *To prove that*

$$C(n, r-1) + C(n, r) = C(n+1, r),$$

for all non-negative integers r and n and $0 \leq r \leq n$.

Proof.

$$\begin{aligned}
C(n, r-1) + C(n, r) &= \tfrac{n!}{(r-1)!(n-r+1)!} + \tfrac{n!}{r!(n-r)!} \\
&= n! \left\{ \tfrac{1}{(r-1)!(n-r+1)!} + \tfrac{1}{r!(n-r)!} \right\} \\
&= \tfrac{n!}{r!(n-r+1)!} \left[\tfrac{r!}{(r-1)!} + \tfrac{(n-r+1)!}{(n-r)!} \right] \\
&= \tfrac{n!}{r!(n-r+1)!} \left[\tfrac{r(r-1)!}{(r-1)!} + \tfrac{(n-r+1)(n-r)!}{(n-r)!} \right] \\
&= \tfrac{n!}{r!(n-r+1)!} [(r + n - r + 1)] \\
&= \tfrac{n!}{r!(n-r+1)} (r + n - r + 1) \\
&= \tfrac{n!}{r!(n-r+1)} (n+1) \\
&= \tfrac{(n+1)n!}{r!(n-r+1)!} \\
&= \tfrac{(n+1)!}{r!(n-r+1)!} \\
&= c(n+1, r)
\end{aligned} \tag{7.3.4}$$

which completes the proof. □

Theorem 7.3.4. *For integers n and r with $1 \leq r \leq n-1$,*

$$C(n, r) = C(n-1, r) + C(n-1, r-1)$$

Proof.

$$\begin{aligned}
C(n-1,r)+C(n-1,r-1) &= \tfrac{(n-1)!}{r!(n-1-r)!}+\tfrac{(n-1)!}{(r-1)!(n-1-r+1)!}\\
&= \tfrac{(n-1)!}{r!(n-1-r)!}+\tfrac{(n-1)!}{(r-1)!(n-r)!}\\
&= (n-1)!\left\{\tfrac{1}{r!(n-1-r)!}+\tfrac{1}{(r-1)!(n-r)!}\right\}\\
&= \tfrac{(n-1)!}{r!(n-r)!}\left[\tfrac{(n-r)!}{(n-1-r)!}+\tfrac{r!}{(r-1)!}\right]\\
&= \tfrac{(n-1)!}{r!(n-r)!}\left[\tfrac{(n-r)(n-1-r)!}{(n-1-r)!}+\tfrac{r(r-1)!}{(r-1)!}\right]\\
&= \tfrac{(n-1)!}{r!(n-r)!}\left[(n-r)+r\right]\\
&= \tfrac{(n-1)!}{r!(n-r)!}(n-r+1)\\
&= \tfrac{(n-1)!}{r!(n-r)!}(n)\\
&= \tfrac{n(n-1)!}{r!(n-r)!}\\
&= \tfrac{n!}{r!(n-r)!}\\
&= C(n,r)
\end{aligned} \tag{7.3.5}$$

□

Theorem 7.3.5. *The total number of ways of making a selection by taking some or all of n objects is given by;*

$$2^n = C(n,0)+C(n,1)+C(n,2)+\ldots+C(n,n)$$

Proof. We can assert this theorem by showing that both sides of the above equation count the number of ways of making selections in n objects. First, we observe that every number of ways of making selection in n objects is an r-selections in n objects for some $r = 0,1,2,\ldots,n$. Since $C(n,r)$ equals the number of r-selections in n, it follows from the addition principle that,

$$C(n,0)+C(n,1)+C(n,2)+\ldots+C(n,n)$$

equals the number of selections in n.

We can also count the number of ways of making the selections in n as follows: Every object can be dealt with in two ways, either it may be selected or left, and since the ways of dealing with one particular object is associated with the ways of dealing with each of the other objects. Therefore, the number of ways of dealing with the n objects by multiplication principle is,

$$2 \times 2 \times 2 \times \ldots \times 2 \text{ to } n \text{ factor } = 2^n.$$

This, however, includes the case in which all the objects are left, which is not permissible. Hence, the required number of sections is,

$$2^n - 1$$

Similarly, from the addition principle, $C(n, 0)$ indicates the case in which no selections is made and should therefore be neglected as well. □

Example 7.3.4

Suppose there are five birds in a nest. In how many ways can one or more of the birds fly out of the nest?

Solution

Applying the last theorem, the required number of ways is,

$$2^5 - 1 = 32 - 1 = 31 \text{ ways}$$

or

$$\begin{aligned} 2^5 &= C(C, 0) + C(5, 1) + C(5, 2) + C(5, 3) + C(5, 4) + C(5, 5) \\ &= 1 + 5 + 10 + 10 + 5 + 1 \\ &= 32 - 1 \\ &= 31 \text{ ways} \end{aligned}$$

7.4 Binomial Theorem

Any expression such as $(a + x)$ involving two terms is known as *Binomial Expression*, and thus $(a + x)^n$ is a Binomial function, while the statement of its expansion in powers of x is known as *Binomial Theorem*; where n is the index of power which could be either rational, positive or negative integer.

The theorem states that,

$$(a+x)^n = \sum_{r=0}^{n} {}^nC_r a^{n-r} x^r$$

where ${}^nC_r$ are called the Binomial Coefficients.

The methods used in proving the binomial theorem for a positive integral index is either by elementary algebraic expansion or by the method of induction; and in this method it is assumed that the expansion holds true for the power of n and subsequently when n is replaced by $(n + 1)$

Proof By Algebra.

$$\begin{aligned}
(a+x)^1 &= a+x \\
(a+x)^2 &= a^2+2ax+x^2 \\
&= a^2+^2C_1ax+x^2 \\
(a+x)^3 &= a^3+3a^2x+3ax^2+x^3 \\
&= a^3+^3C_1a^2x+^3C_2ax^2+x^3 \\
(a+x)^4 &= a^4+4a^3x+6a^2x^2+4ax^3+x^4 \\
&= a^4+^4C_1a^3x+^4C_2a^2x^2+^4C_3ax^3+x^4
\end{aligned}$$

Therefore, the binomial theorem is true for $n = 2, n = 3$ and $n = 4$. And since the theorem is true for $n = 4$, then it is true for $n = 5$ and must be true for $n = 6$, and so on for all positive integer values of n.

Thus for all positive integer values of n,

$$
\begin{aligned}
(a+x)^n &= \textstyle\sum_{x=0}^{n} {}^nC_r a^{n-r}x^r \\
&= a^n +^n C_1 a^{n-1}x +^n C_2 a^{n-2}x^2 + \ldots +^n C_r a^{n-r}x^r \\
&\quad + \ldots + x^n
\end{aligned} \tag{7.4.1}
$$

□

Corollary 7.4.1. *The binomial theorem also applies to the case where n is not a positive integer. This can be achieved by substituting '$-n$' for 'n' as follows:*

$$
\begin{aligned}
(a+x)^{-n} &= \sum_{r=0}^{-n} {}^{-n}C_r a^{-n-r}x^r \\
&= a^{-n} + (-na^{-n-1}x) + \frac{-n(-n-1)}{2!}a^{-n-2}x^2 + \ldots \\
&\quad + \frac{-n(-n-1)(-n-2)\ldots(-n-r+1)}{r!}a^{-n-r}x^r \\
&\quad + \ldots + x^n
\end{aligned} \tag{7.4.2}
$$

However, we shall not be concerned with the proofs when the index is negative or rational number, but it is assumed that the binomial theorem is also valid for n being rational or negative from the above expansion in (7.4.2). Consequently from the algebraic proof, the coefficients in the expansion of the successive powers of $(a + x)$ can be arranged in a triangular array of numbers, called Pascal's triangle, as shown below:

Binomial	*Coefficients*
$(a+x)^1$	1 1
$(a+x)^2$	1 2 1
$(a+x)^3$	1 3 3 1
$(a+x)^4$	1 4 6 4 1
$(a+x)^5$	1 5 10 10 5 1
$(a+x)^6$	1 6 15 20 15 6 1

Proof of the Binomial Theorem By Mathematical Induction for Positive Integer Indices.

$$
\begin{aligned}
(a+x)^n &= \sum_{r=0}^{n} {}^n c_r a^{n-r} x^r \\
&= a^n +^n C_1 a^{n-1}x + {}^nC_2 a^{n-2}x^2 + \ldots + {}^nC_{r-1}a^{n-r+1}x^{r-1} \\
&\quad +^nC_r a^{n-r}x^r + \ldots + x^n \qquad (i)
\end{aligned}
$$

Multiplying equation (i) by $(a + x)$ and showing only necessary terms;

$$
\begin{aligned}
(a+x)^{n+1} &= a^{n+1} + {}^nC_1 a^n x + {}^nC_2 a^{n-1}x^2 + \ldots + {}^nC_{r-1}a^{n-r+1}x^{r-1} \\
&\quad +^nC_r a^{n-r}x^r + \ldots + ax^n + a^n x + {}^nC_1 a^{n-1}x^2 + \\
&\quad \ldots + {}^nC_{r-1}a^{n-r+1}x^r + C_r a^{n-r}x^{r+1} + \ldots + x^{n+1} \qquad (ii)
\end{aligned}
$$

Combining the like terms in powers of x, equation (ii) becomes;

$$
\begin{aligned}
(a+x)^{n+1} &= a^{n+1} + ({}^nC_1 + 1)a^n x + ({}^nC_2 +^n C_1)a^{n-1}x^2 + \ldots \\
&\quad +({}^nC_r +^n C_{r-1})a^{n-r+1}x^r + \ldots + x^{n+1} \qquad (iii)
\end{aligned}
$$

Now,

$$
{}^nC_1 + 1 = n + 1 =^{n+1} C_1 \qquad (iv)
$$

and,

$$
\begin{aligned}
{}^nC_r + {}^nC_{r-1} &= \frac{n!}{(n-r)!r!} + \frac{n!}{(n-r+1)!(r-1)!} \\
&= n!\left\{\frac{1}{(n-r)!r!} + \frac{1}{(n-r+1)!(r-1)!}\right\} \\
&= \frac{n!}{(n-r+1)!r!}\left\{\frac{(n-r+1)!}{(n-r)!} + \frac{r!}{(r-1)!}\right\} \\
&= \frac{n!}{(n-r+1)!r!}\{(n-r+1)+r\} \\
&= \frac{(n+1).n!}{(n-r+1)!r!} \\
&= \frac{(n+1)!}{(n-r+1)!r!} \\
&= {}^{n+r}C_r \qquad (v)
\end{aligned}
$$

Using the results of equation (iv) and (v), equation (iii) becomes;

$$
\begin{aligned}
(a+x)^{n+1} &= \sum_{r=0}^{n+1} {}^{n+1}C_r a^{n+1-r} x^r \\
&= a^{n+1} + {}^{n+1}C_1 a^n x + {}^{n+1}C_2 a^{n-1} x^2 + \\
&\quad \ldots + {}^{n+1}C_r a^{n-r+1} x^r + \ldots + x^{n+1} \qquad (vi)
\end{aligned}
$$

Observe that suppose we replace n by $(n+1)$ in equation (i), the equation (vi) can be easily obtained. Thus the binomial theorem is true for the index n, and must also be true for the index $(n+1)$. □

7.4.1 Properties of Binomial Expansion

(i) When the index n is a positive integer, there are a finite $(n+1)$ terms in the expansion, and in all other cases there are infinite number.

(ii) If the terms are homogeneous, i.e. of degree n, then the sum of the powers of a and x in each term is equal to n

(iii) If the coefficients are symmetrical - the first coefficient is one (i.e ${}^nC_0 = 1$) and the last ${}^nC_n = 1$ as well.

Example 7.4.1 (Finite Series)

Find the Binomial series expansion in ascending powers of x for;
(i) $(2+3x)^4$; (ii) $(3-x)^5$, (iii) $\left(x - \frac{1}{3x}\right)^3$

Solution

(i)

$$
\begin{aligned}
(2+3x)^4 &= \sum_{r=0}^{4} {}^4C_r 2^{4-r} (3x)^r \\
&= 2^4 + {}^4C_1 2^3 (3x) + {}^4C_2 2^2 (3x)^2 + {}^4C_3 2 (3x)^3 + (3x)^4 \\
&= 16 + 96x + 216x^2 + 216x^3 + 81x^4
\end{aligned}
$$

(ii)

$$
\begin{aligned}
(3-x)^5 &= \sum_{r=0}^{5} {}^5C_r 3^{5-r}(-x)^r \\
&= 3^5 + {}^5C_1 3^4(-x) + {}^5C_2 3^3(-x)^2 \\
&\quad + {}^5C_3 3^2(-x)^3 + {}^5C_4 3(-x)^4 + (-x)^5 \\
&= 243 - 405x + 270x^2 - 90x^3 + 15x^4 - x^5
\end{aligned}
$$

(iii)

$$
\begin{aligned}
\left(x - \frac{1}{3x}\right)^3 &= \sum_{r=0}^{3} {}^3C_r x^{3-r}\left(\frac{1}{-3x}\right)^r \\
&= x^3 + {}^3C_1 x^2\left(\frac{1}{-3x}\right) + {}^3C_2 x\left(\frac{1}{-3x}\right)^2 \\
&\quad + {}^3C_3 x^0\left(-\frac{1}{3x}\right)^3 \\
&= x^3 - \frac{3x^2}{3x} + \frac{3x}{9x^2} - \frac{1}{27x^3} \\
&= x^3 - x + \frac{1}{3x} - \frac{1}{27x^3}
\end{aligned}
$$

Remark 16 *In the expansion* $(a+x)^n$*; if* $a = 1$*, then the expansion becomes;*

$$
\begin{aligned}
(1+Z)^r &= 1 + nZ + n\frac{(n-1)}{2!}Z^2 + \dots \\
&\quad + \frac{n(n-1)(n-2)\dots(n-r+1)}{r!}Z^r + \dots
\end{aligned}
$$

which is easier to manipulate than the general expansion. i.e.,

$$\frac{n(n-1)(n-2)\dots(n-r+1)}{r!}a^{n-r}Z^r; \quad |Z| < 1$$

which is the $(r+1)^{th}$ *term in the binomial expansion.*

Hence, it is instructive in many cases to base the expansion of $(a+x)^n$ *on the expansion of* $(1+z)^n$ *as follows;*

$$(a+x)^n = [a(1+x/a)]^n = a^n(1+Z)^n \qquad (7.4.2)$$

where,

$$Z = \frac{x}{a}$$

Remark 17

$$2^n = (1+1)^n = \sum_{r=o}^{n} {}^nC_r \tag{7.4.3}$$

i.e. the sum of the numbers on the n^{th} term is 2^n.

Example 7.4.2 (Infinite Series)

Find the Binomial series expansion in ascending powers of x for the 1^{st} 4 terms;
(i) $\sqrt{4+5x}$; (ii) $(1+x)^{1/3}$; (iii) $\frac{1}{3-x}$; (iv) $\frac{1}{(2+x)^3}$;
(v) $\frac{1}{\sqrt[3]{8-3x}}$

Solution

(i)

$$\begin{aligned}\sqrt{4+5x} &= (4+5x)^{1/2} \\ &= [4(1+\frac{5x}{4})]^{1/2} \\ &= 2(1+\frac{5x}{4})^{1/2}\end{aligned}$$

Thus,

$$\begin{aligned}2(1+\frac{5}{4}x)^{1/2} &= 2\left\{1+\frac{1}{2}\left(\frac{5}{4}x\right)+\frac{\frac{1}{2}(\frac{1}{2}-1)}{2!}\left(\frac{5}{4}x\right)^2\right. \\ &\quad \left. +\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}\left(\frac{5}{4}x\right)^3+\ldots\right\} \\ &= 2\left\{1+\frac{5}{8}x-\frac{25}{128}x^2+\frac{125}{1024}x^3+\ldots\right\}\end{aligned}$$

(iv)

$$\begin{aligned}\frac{1}{(2+x)^3} &= \frac{1}{[2(1+\frac{x}{2})]^3}\\ &= \frac{1}{2^3(1+\frac{x}{2})^3}\end{aligned}$$

Thus,

$$\begin{aligned}2^{-3}(1+\frac{x}{2})^{-3} &= \frac{1}{8}\left\{1+[-3(\frac{x}{2})]+\frac{(-3)(-3-1)}{2!}(\frac{x}{2})^2\right.\\ &\left.+\frac{(-3)(-3-1)(-3-2)}{3!}(\frac{x}{2})^3+\ldots\right\}\\ &= \frac{1}{8}\left\{1-\frac{3}{2}x+\frac{12}{2}\left(\frac{x^2}{4}\right)-\frac{60}{6}\left(\frac{x^3}{8}\right)+\ldots\right\}\\ &= \frac{1}{8}\left\{1-\frac{3}{2}x+\frac{3}{2}x^2-\frac{5}{4}x^3+\ldots\right\}\end{aligned}$$

(v)

$$\begin{aligned}\frac{1}{\sqrt[3]{8-3x}} &= \frac{1}{[8(1-\frac{3x}{8})]^{1/3}}\\ &= \frac{1}{8^{1/3}(1-\frac{3x}{8})^{1/3}}\end{aligned}$$

Thus,

$$\begin{aligned}\frac{1}{2}\left(1-\frac{3x}{8}\right)^{-1/3} &= \frac{1}{2}\left\{1+[-\frac{1}{3}(\frac{-3x}{8})]\right.\\ &+\frac{-\frac{1}{3}(-\frac{1}{3}-1)}{2!}\left(\frac{-3x}{8}\right)^2\\ &\left.+\frac{-\frac{1}{3}(-\frac{1}{3}-1)(-\frac{1}{3}-2)}{3!}(\frac{-3x}{8})^3+\ldots\right\}\\ &= \frac{1}{2}\left\{1+\frac{3}{24}x-\frac{2}{9}.\frac{9x^2}{64}+\frac{10}{81}.\frac{27x^3}{512}+\ldots\right\}\\ &= \frac{1}{2}\left\{1+\frac{1}{8}x-\frac{x^2}{32}+\frac{5}{768}x^3+\ldots\right\}\end{aligned}$$

Questions (ii) and (iii) are left as exercises.

Remark 18 *The following expansions, which can be readily obtained by using the binomial theorem should be studied thoroughly;*

$$\begin{aligned}
\frac{1}{1-x} &= (1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots + x^n + \ldots \\
\frac{1}{1+x} &= (1+x)^{-1} = 1 - x + x^2 - x^3 + \ldots \\
\frac{1}{(1-x)^2} &= (1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \ldots \\
\frac{1}{(1+x)^2} &= (1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \ldots
\end{aligned}$$

In each the above cases, the expansion is only true if $-1 < x < 1$, *i.e.* $|x| < 1$.

7.4.2 Application of Binomial Expansion in Approximation

Suppose x is small relative to a in the binomial function $(a + x)^n$, then the approximation chosen in any particular problem is dependent upon the degree of accuracy required in the result. We shall illustrate with the following examples.

Example 7.4.3

Use Binomial series expansion to compute the following to four decimal places. (i) $(0.98)^4$; (ii) $(1.98)^4$; (iii) $\frac{1}{99}$; (iv) $\sqrt{9.018}$

Solution

(i) Write 0.98 as $1 - 0.02$, then we have;

$$(0.98)^4 \equiv (1 - 0.02)^4$$

Thus,

$$\begin{aligned}(1-0.02)^4 &= 1+{}^4C_1(-0.02)+{}^4C_2(-0.02)^2\\ &\quad +{}^4C_3(-0.02)^3+(-0.02)^4\\ &= 1-0.08+0.002\,4-0.000\,032\\ &\quad +0.000\,000\,16\\ &= 0.922\,368\,1\\ &\simeq 0.922\,4.\end{aligned}$$

(ii) $(1.98)^4 \equiv (2-0.02)^4 = 2^4(1-0.01)^4$
Thus,

$$\begin{aligned}2^4(1-0.001)^4 &= 2^4[1+{}^4C_1(-0.01)+{}^4C_2(-0.01)^2\\ &\quad +{}^4C_3(-0.01)^3+(-0.01)^4]\\ &= 2^4[1-0.04+0.000\,6-0.000\,004\\ &\quad +0.000\,000\,01]\\ &= 15.369\,5\end{aligned}$$

(iii)

$$\begin{aligned}\frac{1}{99}\equiv\frac{1}{(100-1)} &= \frac{1}{100(1-\frac{1}{100})}\\ &= \frac{1}{100}(1-0.01)^{-1}\end{aligned}$$

Thus,

$$\begin{aligned}\frac{1}{100}(1-0.01)^{-1} &= \frac{1}{100}[1+0.01+(0.01)^2+(0.01)^3+\ldots]\\ &= \frac{1}{100}(1.010\,101)\\ &= 0.0101\end{aligned}$$

(iv) $\sqrt{9.018} \equiv (9.018)^{1/2} = [9(1.002)]^{1/2} = 3(1+0.002)^{1/2}$

Thus,

$$\begin{aligned}
3(1+0.002)^{1/2} &= 3\left[1+\frac{1}{2}(0.002)+\frac{\frac{1}{2}(\frac{1}{2}-1)}{2!}(0.002)^2\right.\\
&\quad \left.+\frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{3!}.(0.002)^3+\ldots\right]\\
&= 3(1+0.001-0.000\,000\,5+0.000\,000\,000\,5\\
&\quad +\ldots)\\
&= 3(1.000\,999\,5)\\
&= 3.002\,998\,5\\
&\simeq 3.003\,0
\end{aligned}$$

Example 7.4.4

Find the 1st three terms in the expansion of,

$$\frac{(1+x)^{\frac{2}{3}}+\sqrt{1+5x}}{(1+\frac{2}{3}x)^{-3}(4-5x)^{\frac{1}{2}}}$$

Solution

$$\frac{(1+x)^{\frac{2}{3}}+\sqrt{1+5x}}{(1+\frac{2}{3}x)^{-3}(4-5x)^{\frac{1}{2}}}) = (1+x)^{\frac{2}{3}}+(1+5x)^{\frac{1}{2}}(1+\frac{2}{3}x)^3(4^{\frac{-1}{2}})(1-\frac{5x}{4})^{-\frac{1}{2}}$$

$$\begin{aligned}
&= \left\{1+\frac{\frac{2}{3}}{1!}(x)+\frac{\frac{2}{3}(\frac{2}{3}-1)}{2!}x^2+\ldots\right\}\\
&\quad \times\left\{1+\frac{\frac{1}{2}}{1!}(5x)+\frac{\frac{1}{2}(\frac{1}{2}-1)}{2!}(5x)^2+\ldots\right\}\\
&\quad \times\left\{\left[1+\frac{3}{1!}(\frac{2}{3}x)+\frac{3(3-1)}{2!}(\frac{2}{3}x)^2+\ldots\right]\right.\\
&\quad \left.\times\frac{1}{2}\left[1+\frac{-\frac{1}{2}}{1!}(\frac{5}{4}x)+\frac{-\frac{1}{2}(-\frac{1}{2}-1)}{2!}(\frac{5}{4}x)^2+\ldots\right]\right\}
\end{aligned}$$

$$
\begin{aligned}
&= \left\{1+\frac{2}{3}x-\frac{1}{9}x^2+1+\frac{5}{2}x-\frac{25}{8}x^2+\ldots\right\} \\
&\quad \times\frac{1}{2}\left\{\left(1+2x+\frac{4}{3}x^2+\ldots\right)\left(1-\frac{5}{8}x-\frac{75}{128}x^2+\ldots\right)\right\} \\
&= \left(2+\frac{19}{6}x-\frac{233}{72}x^2+\ldots\right)\times\frac{1}{2}\left[\left(1+2x+\frac{4}{3}x^2+\ldots\right)\right. \\
&\quad \left.+\left(\frac{-5}{8}x-\frac{5}{4}x^2+\ldots\right)+\left(\frac{-75}{128}x^2+\ldots\right)\right] \\
&= \left(2+\frac{19}{6}x-\frac{233}{72}x^2+\ldots\right)\times\frac{1}{2}\left(1+\frac{11}{8}x-\frac{193}{384}x^2+\ldots\right) \\
&= \left(2+\frac{19}{6}x-\frac{233}{72}x^2+\ldots\right)\left(2+\frac{22}{8}x-\frac{193}{192}x^2+\ldots\right) \\
&= \left(4+\frac{19}{3}x-\frac{233}{36}x^2+\ldots\right)+\left(\frac{11}{2}x+\frac{209}{24}x^2+\ldots\right) \\
&\quad +\left(\frac{-193}{96}x^2+\ldots\right) \\
&= 4+\frac{71}{6}x+\frac{65}{288}x^2+\ldots \\
&= 4+\frac{71}{6}x+\frac{65}{288}x^2
\end{aligned}
$$

7.4.3 Extension of the Binomial Theorems of the Type $(a+b+c)^n$ etc

The binomial theorem can be extended to expression containing three terms such as $(a+b+c)^n$ referred to as *trinomial expression* and further to *matrimonial expression* containing more than three terms. But for the purpose of this course, we shall restrict our emphasis to trinomial expression only, which is of the type $(a+b+c)^n$. In this case the three terms are grouped into two, with each containing one and two terms respectively; while each quantity are treated as a single term initially.

Consequently, the expression $(a+b+c)^n$ can be grouped into two quantities, a and $(b+c)$ respectively, and the expression to be expanded will be of the form;

$$\begin{aligned}[a+(b+c)]^n &= \sum_{r=0}^{n} {}^nC_r[a^{n-r}(b+c)^r] \\ &= a^n + na^{n-1}(b+c) + \frac{n(n-1)}{2!}a^{n-2}(b+c)^2 \\ &\quad + \frac{n(n-1)(n-2)}{3!}a^{n-3}(b+c)^3 + \dots \end{aligned} \tag{7.4.4}$$

The expansion of the equalities $(b+c)^2, (b+c)^3$, etc., can subsequently be obtained using binomial theorem.

Remark 19 *If any of the terms a, b or c is equal to unity, for simplification, that term is generally treated as a single term while the remaining two is combined as a single term.*

Example 7.4.5

Expand $(2-3x-2x^2)^5$ in ascending powers of x as far as the term in x^4.

Solution

$$\begin{aligned}(2-3x-2x^2)^5 &= 2^5(1-\frac{3x}{2}-x^2)^5 \\ &= 2^5[1-x(\frac{3}{2}+x)]^5\end{aligned}$$

$$
\begin{aligned}
&= 2^5\left\{1+\frac{5}{1!}\left[-x(\frac{3}{2}+x)\right]+\frac{5(5-1)}{2!}\left[-x\left(\frac{3}{2}+x\right)\right]^2+\dots\right\}\\
&= 2^5\left\{1-5x\left(\frac{3}{2}+x\right)+10x^2\left(\frac{3}{2}+x\right)^2+\dots\right\}\\
&= 2^5\left\{1-\frac{15x}{2}-5x^2+10x^2\left(\frac{9}{4}+3x+x^2\right)+\dots\right\}\\
&= 2^5\left\{1-\frac{15x}{2}-5x^2+\frac{45x^2}{2}+30x^3+10x^4+\dots\right\}\\
&= 2^5\left\{1-\frac{15x}{2}+\frac{35}{2}x^2+30x^3+10x^4+\dots\right\}\\
&= 1-240x+560x^2+960x^3+320x^4+\dots
\end{aligned}
$$

Example 7.4.6

If $(1+mx+nx^2)(1-2x)^{18}$ can be expressed in ascending powers of x, determine m and n if the coefficients of x^3 and x^4 are both zero.

Solution

$$
\begin{aligned}
(1+mx+nx^2)(1-2x)^{18} = {} & (1+mx+nx^2)\left[1+\frac{18}{1!}(-2x)\right.\\
&+\frac{18(18-1)}{2!}(-2x)^2\\
&+\frac{18(18-1)(18-2)}{3!}(-2x)^3\\
&+\frac{18(18-1)(18-2)(18-3)}{4!}(-2x)^4\\
&+\dots]
\end{aligned}
$$

$$\begin{aligned}
&= (1+mx+nx^2)(1-36x+612x^2-6528x^3+48960x^4+\ldots)\\
&= (1+mx+nx^2)+(-36x-36mx^2-36nx^3)\\
&\quad +(612x^2+612mx^3+612nx^4)+(-6528x^3-6528mx^4+\ldots)\\
&\quad +48960x^4+\ldots\\
&= 1+(m-36)x+(n-36m+612)x^2+(-36n\\
&\quad +612m-6528)x^3+(612n-6528m+48960)x^4+\ldots
\end{aligned}$$

It follows that, if the coefficient of x^3 and x^4 are both zero, then;

$$\begin{aligned}
-36n+612m-6528 &= 0\\
\text{i.e.,}\quad -36n+612m &= 6\,528 \qquad (i)
\end{aligned}$$

and,

$$\begin{aligned}
612n+48960-6528m &= 0\\
\text{i.e.,}\quad 612n-6528m &= -48\,960 \qquad (ii)
\end{aligned}$$

Now, solving equations (i) and (ii) simultaneously we obtain;

$$\begin{aligned}
-36n+612m &= 6528(17)\\
612n-6528m &= -48\,960\\
\\
-612n+10404m &= 1\,100\,976\\
612n-6528m &= -48\,960\\
\\
\text{i.e.,}\quad 3876m &= 62\,016\\
\therefore\quad m &= \frac{62016}{3876}=16
\end{aligned}$$

By substituting for m in equation (i) we have;

$$\begin{aligned}
-36n+612(16) &= 6\,528\\
-36n &= -9792+6528\\
&= -3264\\
n &= \frac{3264}{36}\\
&= 90(\frac{3}{4})\\
\therefore\quad m &= 16 \text{ and } n=90\frac{3}{4}
\end{aligned}$$

Chapter 8

Elementary Trigonometry

Trigonometry is the branch of mathematics that deals with the study of the relations of sides and angles of triangles, and of the methods of applying these relations in the solutions of problems involving triangles. This however, does not mean the elementary angle measurement of plane geometry; in which the magnitude of the angle is read off on a protractor, but calculation with special functions that depend on angles which are called trigonometric functions. The trigonometric function is a function of an angle expressed as a ratio of two of the sides of a right triangle that contains the angle. Trigonometry has wide applications in Land Surveying, Urban and Regional Planning, Airspace Management, Navigation Systems etc.

8.1 Circular Functions of the General Angle

In elementary mathematics, the definition of the trigonometric ratios sine, cosine and tangent are defined in terms of the angle θ; for $0 \leq \theta \leq 360°$. These definitions can be extended to angles of any magnitude, the general angle, which could be either positive or negative.

Let OX, and OY be the usual perpendicular axes in the x, y-coordinate system; and suppose that $P(x, y)$ is a point i.e. $OP = r$, and $\angle POX = \theta$, as in Figure 8.1.

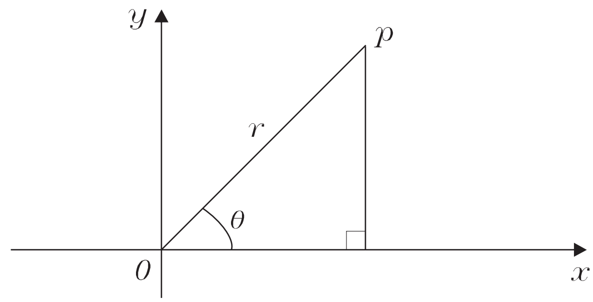

Figure 8.1: General Angle

then we have;

$$\sin\theta = \frac{y}{r}; \quad \cos\theta = \tfrac{x}{r}, \quad \tan\theta = \frac{y}{x}$$
$$\operatorname{cosec}\theta = \frac{r}{y}; \quad \sec\theta = \tfrac{r}{x}, \quad \cot\theta = \frac{x}{y}$$

It is to be noted that the ratios may be positive or negative, depending on which the quadrant the arm OP lies in, and the values of the ratios can be related to those of the corresponding acute angle.

(i) **Obtuse angles;** $(90^\circ \leq \theta \leq 180^\circ)$

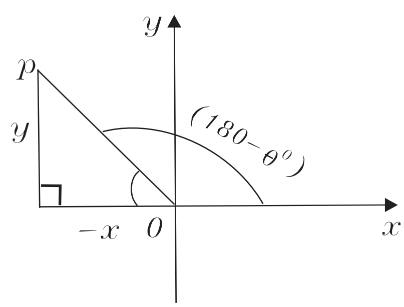

Figure 8.2: Obtuse Angle

$$\begin{aligned}\sin(180-\theta)^\circ &= \sin\theta^\circ \\ \cos(180-\theta)^\circ &= -\cos\theta^\circ \\ \tan(180-\theta)^\circ &= -\tan\theta^\circ\end{aligned}$$

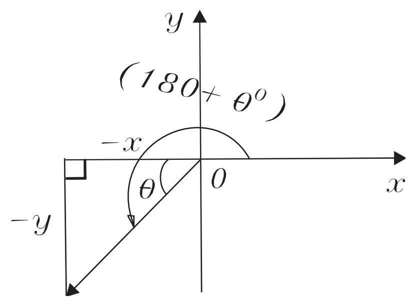

Figure 8.3: Reflex Angle

(ii) **Reflex angles;** $(180° \leq \theta \leq 270°)$

$$\begin{aligned} \sin(180+\theta)^\circ &= -\sin\theta^\circ \\ \cos(180+\theta)^\circ &= -\cos\theta^\theta \\ \tan(180+\theta)^\circ &= \tan\theta^\circ \end{aligned}$$

(iii) $270° \leq \theta \leq 360°$

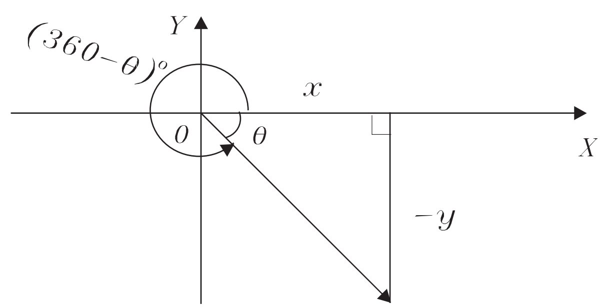

Figure 8.4: Angle: $270° \leq \theta \leq 360°$

$$\begin{aligned} \sin(360-\theta)^\circ &= -\sin\theta^\circ \\ \cos(360-\theta)^\circ &= \cos\theta^\circ \\ \tan(360-\theta)^\circ &= \tan\theta^\circ \end{aligned}$$

In summary, the signs of the circular functions of θ in the four quadrants are displayed in the following table.

Quadrant	x	y	r	$\sin\theta$	$\cos\theta$	$\tan\theta$
1st	+	+	+	+	+	+
2nd	-	+	+	+	-	-
3rd	-	-	+	-	-	+
4th	+	-	+	-	+	-

Table 8.1: Signs of Circular Functions of θ in the Four Quadrants

However, these results could be put in a shorthand form using the word "CAST" to show what functions are positive in the various quadrants.

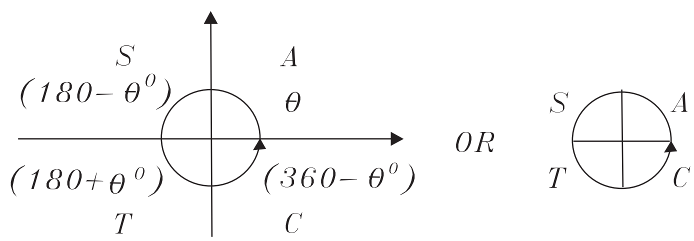

Figure 8.5: Shorthand Form of Various Quadrants

Where (A= all, S = sin, T = tan, and C = cos)

Observe that the signs for the reciprocals of these functions;

$$\operatorname{cosec}\theta = \frac{1}{\sin\theta}; \quad \sec\theta = \frac{1}{\cos\theta}; \quad \cot\theta = \frac{1}{\tan\theta}$$

can be obtained from the above table.

Example 8.1.1

Find the values of the sine, cosine and tangent of the following angles:
(i) 130° (ii) 250° (iii) 300° (iv) 460°

Solution

(i) $\sin 130° = \sin(180 - 130)° = \sin 50° = 0.766\ 0$
$\cos 130 = -\cos(180 - 130)° = -\cos 50° = -0.642\ 8$
$\tan 130° = -\tan(180 - 130)° = \tan 50° = -1.191\ 8$

(ii) $\sin 250° = -\sin(250 - 180)° = -\sin 70° = -0.939\,7$
$\cos 250° = -\cos(250 - 180)° = -\cos 70° = -0.3420$
$\tan 250° = \tan(250 - 180)° = \tan 70° = 2.7475$

(iii) $\sin 300° = -\sin(360 - 300)° = -\sin 60° = -0.866\,0$
$\cos 300° = \cos(360 - 300)° = \cos 60° = 0.500\,0$
$\tan 300° = -\tan(360 - 300)° = -\tan 60° = -1.732\,1$

(iv) $460° = 360° + 100°$
$\sin 460° = \sin 100° = \sin(180 - 100)° = \sin 80° = 0.984\,9$
$\cos 460° = \cos 100° = -\cos 80° = -0.1734$
$\tan 460° = \tan 100° = -\tan 80° = -5.671\,3$

In general, if α is an angle such that,

$$\alpha = n \times 360° + \theta$$

where n is a positive integer, and $\theta° \leq \theta \leq 360°$, as in Example 8.1.1 (iv), then;

$$\begin{aligned} \sin\alpha &= \sin\theta \\ \cos\alpha &= \cos\theta \\ \tan\alpha &= \tan\theta \end{aligned}$$

8.1.1 Special Angles

Usually, the ratios for angles $0°, 30°, 45°, 60°, 90°, 180°, 270°$ occur frequently in real life problems, and it is useful to remember their values. Their sine, cosine, tangent can be derived in surd form from single right-angled triangles as follows:

Angles $0°$ **and** $90°$

Let $\triangle ABC$ be right-angle at B, and let the angle A be small. Then as angle A approaches zero, angle C approaches $90°$. Consequently, the lengths AB approaches AC and BC approaches zero.

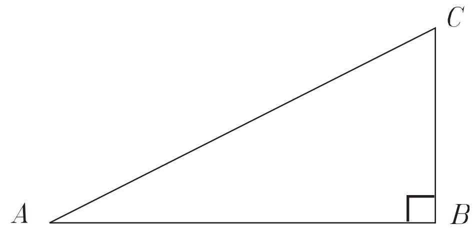

Figure 8.6: Right-angle triangle

From figure 8.6, the following limits exists:

$$\begin{aligned}
\sin 0^\circ &= \lim_{\angle A\to 0} \frac{BC}{AC} = \frac{0}{AC} = 0\\
\cos 0^\circ &= \lim_{\angle A\to 0} \frac{AB}{AC} = \frac{AC}{AC} = 1\\
\tan 0^\circ &= \lim_{\angle A\to 0} \frac{BC}{AB} = \frac{0}{AB} = 0
\end{aligned}$$

Similarly,

$$\begin{aligned}
\sin 90^\circ &= \lim_{\angle A\to 0} \frac{AB}{AC} = \frac{AC}{AC} = 1\\
\cos 90^\circ &= \lim_{\angle A\to 0} \frac{BC}{AC} = \frac{0}{AC} = 0\\
\tan 90^\circ &= \lim_{\angle A\to 0} \frac{AB}{BC} = \frac{AB}{0} \ \text{(undefined)}
\end{aligned}$$

Angles 30° **and** 60°

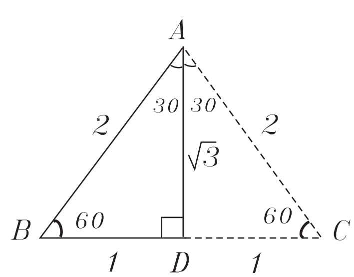

Figure 8.7: Angles 30° and 60°

Figure 8.7 is an equilateral triangle ABC, of side 2 units. Suppose AD bisects angle A, then by symmetry, it is perpendicular to BC. Hence $BD = 1$ and by Pythagoras theorem;

$$\begin{aligned} AD^2 &= AB^2 - BD^2 \\ &= 2^2 - 1^2 \\ &= 3 \\ \therefore \quad AD &= \sqrt{3} \\ \sin 30^\circ &= \cos 60^\circ = \frac{1}{2} \\ \cos 30^\circ &= \sin 60^\circ = \frac{\sqrt{3}}{2} \\ \tan 30^\circ &= \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} \\ \tan 60^\circ &= \sqrt{3} \end{aligned}$$

Angle 45°

Consider the isosceles triangle ABC right-angled at C. Suppose $\angle A = \angle B = 45^\circ$ and the sides $AB = AC = 1$ unit, then;

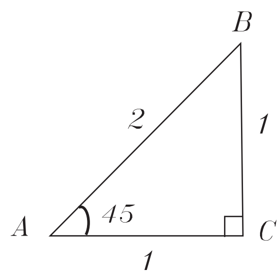

Figure 8.8: Angle 45°

by Pythagoras, $AC = \sqrt{2}$

$$\begin{aligned} \sin 45^\circ = \cos 45^\circ &= \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \\ \tan 45^\circ &= 1 \end{aligned}$$

Angle $180°$

$$\begin{aligned}
\sin 180° &= \sin(180-180)° = \sin 0° = 0 \\
\cos 180° &= -\cos(180-180)° = -\cos 0° = -1 \\
\tan 180° &= -\tan(180-180)° = -\tan 0° = 0
\end{aligned}$$

Angle $270°$

$$\begin{aligned}
\sin 270° &= -\sin(270-180)° = -\sin 90° = -1 \\
\cos 270° &= -\cos(270-180)° = -\cos 90° = 0 \\
\tan 270° &= \tan(270-180)° = \tan 90° \text{ is undefined}
\end{aligned}$$

The following table gives the summary of the values of the special angles:

Angle	$0°$	$30°$	$45°$	$60°$	$90°$	$180°$	$270°$
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	0	-1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	-1	0
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	–	0	–

Table 8.2: Summary of Special Angles

Negative Angles

Observe that from Figure 8.9, if the arm OP rotates in a clockwise direction from x-axis, it will describe a negative angle, $-\theta$.

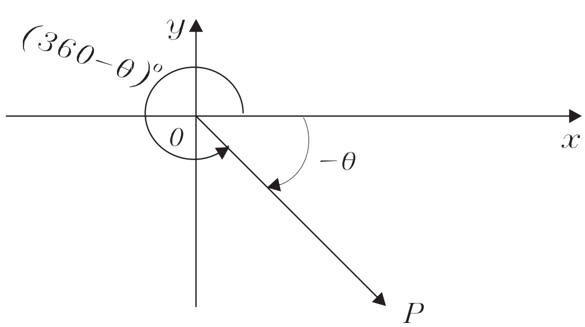

Figure 8.9: Negative Angles

The positive angle is then $(360° - \theta)$.

$$
\begin{aligned}
\therefore \quad \sin(-\theta) &= \sin(360° - \theta) = -\sin\theta \\
\cos(-\theta) &= \cos(360° - \theta) = \cos\theta \\
\tan(-\theta) &= \tan(360° - \theta) = -\tan\theta
\end{aligned}
$$

Example 8.1.2

Find the values of the sine, cosine and tangent of the following angles:
(i) $-120°$ (ii) $-240°$.

Solution

(i)

$$
\begin{aligned}
\sin(-120°) &= \sin(360 - 120)° \\
&= -\sin 120° \\
&= -\sin(180 - 120)° \\
&= -\sin 60° \\
&= -\frac{\sqrt{3}}{2}
\end{aligned}
$$

$$
\begin{aligned}
\cos(-120°) &= \cos(360 - 120)° \\
&= \cos 120° \\
&= \cos(180 - 120)° \\
&= -\cos 60° \\
&= -\frac{1}{2}
\end{aligned}
$$

$$
\begin{aligned}
\tan(-120°) &= \tan(360 - 120)° \\
&= -\tan 120° \\
&= -\tan(180 - 120)° \\
&= \tan 60° \\
&= \sqrt{3}
\end{aligned}
$$

(ii)

$$\begin{aligned}\sin(-240^\circ) &= \sin(360-240)^\circ \\ &= -\sin 240^\circ \\ &= -\sin(240-180)^\circ \\ &= \sin 60 \\ &= \frac{\sqrt{3}}{2}\end{aligned}$$

$$\begin{aligned}\cos(-240^\circ) &= \cos(360-240)^\circ \\ &= \cos 240^\circ \\ &= \cos(240-180)^\circ \\ &= -\cos 60^\circ \\ &= -\frac{1}{2}\end{aligned}$$

$$\begin{aligned}\tan(-240^\circ) &= \tan(360-240)^\circ \\ &= -\tan 240^\circ \\ &= -\tan(240-180)^\circ \\ &= -\tan 60^\circ \\ &= -\sqrt{3}\end{aligned}$$

8.2 Trigonometric Identities

An identity is a mathematical expression that is true for all values of the variable (or variables) contained in it (or when by simplification, both sides are identical), (Section 3.1). In other words, the expressions are identical in value, but look different. For example, the expression $x^2 - y^2 = (x-y)(x+y)$ is an identity, since it is true for all values of x and y. Conversely, the expression $x^2 = 1$ is not an identity since it is only true for $x = \pm 1$

Consider the right-angled triangle ABC (Figure 8.10), with the usual notation, then for any angle θ, the following identities hold:

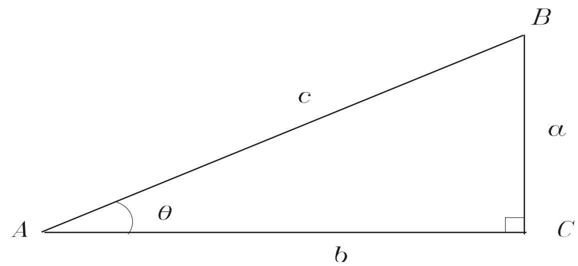

Figure 8.10: Right-angle triangle ABC

(i) $\sin^2\theta + \cos^2\theta = 1$, (Note $\sin^2\theta = (\sin\theta)^2$ etc.)
Now, from Figure 8.10,

$$\begin{aligned} a^2 + b^2 &= c^2 \text{ (by Pythagoras)} \\ \text{i.e., } \sqrt{a^2 + b^2} &= c \end{aligned}$$

and,

$$\sin\theta = \frac{a}{c}, \quad \cos\theta = \frac{b}{c}$$

Squaring both sides, we have;

$$\begin{aligned} (\sin\theta)^2 &= \left(\frac{a}{c}\right)^2 \text{ and } (\cos\theta)^2 = \left(\frac{b}{c}\right)^2 \\ \text{i.e. } \sin^2\theta &= \frac{a^2}{c^2} \text{ and } \cos^2\theta = \frac{b^2}{c^2} \end{aligned}$$

Adding, we have;

$$\begin{aligned} \sin^2 + \cos^2\theta &= \frac{a^2}{c^2} + \frac{b^2}{c^2} \\ &= \frac{a^2 + b^2}{c^2} \\ &= \frac{c^2}{c^2} = 1 \end{aligned}$$

$$\therefore \quad \sin^2\theta + \cos^2\theta = 1 \tag{8.2.1}$$

This relation (8.2.2) is true for whatever value of θ is taken.
Dividing both sides of this identity by $\cos^2\theta$, and recalling that,

$$\frac{\sin\theta}{\cos\theta} = \tan\theta, \quad \frac{1}{\cos\theta} = \sec\theta, \quad \frac{1}{\sin\theta} = \cos ec\theta, \quad \frac{\cos\theta}{\sin\theta} = \cot\theta$$

we have;

$$\begin{aligned}
\frac{\sin^2\theta+\cos^2\theta}{\cos^2\theta} &= \frac{1}{\cos^2\theta}\\
\frac{\sin^2\theta}{\cos^2\theta}+\frac{\cos^2\theta}{\cos^2\theta} &= \frac{1}{\cos^2\theta}\\
\frac{\sin^2\theta}{\cos^2\theta}+1 &= \frac{1}{\cos^2\theta}\\
\left(\frac{\sin\theta}{\cos\theta}\right)^2+1 &= \left(\frac{1}{\cos\theta}\right)^2\\
(\tan\theta)^2+1 &= (\sec\theta)^2\\
\tan^2\theta+1 &= \sec^2\theta
\end{aligned}$$

$$\therefore \qquad 1+\tan^2\theta=\sec^2\theta \tag{8.2.2}$$

which again is true for all values of θ.
Similarly, dividing both sides of (8.2.1) by $\sin^2\theta$, we obtain;

$$\begin{aligned}
\frac{\sin^2\theta+\cos^2\theta}{\sin^2\theta} &= \frac{1}{\sin^2\theta}\\
\frac{\sin^2\theta}{\sin^2\theta}+\frac{\cos^2\theta}{\sin^2\theta} &= \frac{1}{\sin^2\theta}\\
1+\left(\frac{\cos\theta}{\sin\theta}\right)^2 &= \left(\frac{1}{\sin\theta}\right)^2\\
1+(\cot\theta)^2 &= (\cos ec\theta)^2
\end{aligned}$$

$$\therefore \qquad 1+\cot\theta=\cos ec^2\theta \tag{8.2.3}$$

Example 8.2.1

Without using tables and calculator, find the values of the following functions:

(i) $\sin\theta$ and $\cos\theta$, if $\tan\theta=\frac{4}{3}$ and θ is an acute angle.

(ii) $\sin\theta$ and $\tan\theta$, given that $\cos\theta=-\frac{4}{5}$ and $180° < \theta < 270°$.

Solution

(i) Recall that, $1 + \tan^2\theta = \sec^2\theta$
So,

$$\begin{aligned} 1 + \left(\frac{4}{3}\right)^2 &= \frac{1}{\cos^2\theta} \\ \frac{25}{9} &= \frac{1}{\cos^2\theta} \quad \text{or} \quad \cos\theta = \pm\frac{3}{5} \end{aligned}$$

But, θ lies in the 1^{st} quadrant, so that;

$$\cos\theta = \frac{3}{5}$$

Similarly from,

$$\begin{aligned} \tan\theta &= \frac{\sin\theta}{\cos\theta} \\ \frac{4}{3} &= \frac{\sin\theta}{\frac{3}{5}} \\ \sin\theta &= \frac{4}{5} \end{aligned}$$

or diagrammatically;

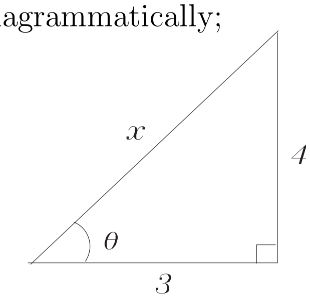

Figure 8.11:

From the Figure 8.11 , we have;

$$\tan\theta = \frac{4}{3} \quad \text{and} \quad \sin\theta = \frac{4}{x}$$

But, $x = 5$, hence we have;

$$\sin\theta = \frac{4}{5} \quad \text{and} \quad \cos\theta = \frac{3}{5}$$

(ii) From the identity: $\sin^2\theta + \cos^2\theta = 1$, we have;

$$\begin{aligned}\sin^2\theta + \left(-\frac{4}{5}\right)^2 &= 1\\ \sin^2\theta &= \frac{9}{25}\\ \sin\theta &= \pm\frac{3}{5}\end{aligned}$$

But, θ lies in the 3^{rd} quadrant, and hence $\sin\theta = -\frac{3}{5}$.
Similarly, from the identity: $\tan\theta = \frac{\sin\theta}{\cos\theta}$, we have;

$$\begin{aligned}\tan\theta &= \frac{-3/5}{-4/5}\\ &= \frac{3}{4}\end{aligned}$$

Alternatively, it could be solved diagrammatically as follows:

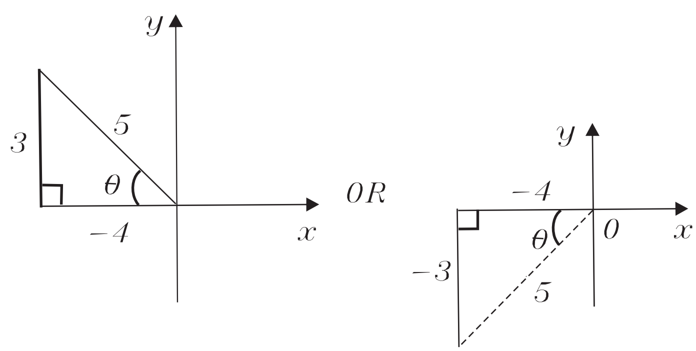

Figure 8.12:

$$\cos\theta = \frac{-4}{5}; \quad \sin\theta = \frac{-3}{5}; \quad \tan\theta = \frac{3}{4}$$

Example 8.2.2

Find the general solution of the equation and hence find θ, given that;
$4\cos^2\theta + 5\sin\theta - 5 = 0$.

Solution

Using $\cos^2\theta = 1 - \sin^2\theta$, the equation becomes;

$$\begin{aligned} 4 - 4\sin^2\theta + 5\sin\theta - 5 &= 0 \\ \text{i.e.,}\quad 4\sin^2\theta - 5\sin\theta + 1 &= 0 \end{aligned}$$

which is a quadratic equation in terms of $\sin\theta$, hence solving quadratically, we have;

$$(4\sin\theta - 1)(\sin\theta - 1) = 0$$

which gives;

$$\begin{aligned} \sin\theta &= \frac{1}{4} && (i) \\ \text{or}\quad \sin\theta &= 1 && (ii) \end{aligned}$$

$$\therefore \quad \left.\begin{aligned} &\text{From } (i), \quad \theta = m.360^\circ \pm 14.48^\circ \\ &\text{From } (ii), \quad \theta = n.360^\circ \pm 90^\circ \end{aligned}\right\}$$

where m and n are any integers.

Example 8.2.3

Prove that for all values of θ;

$$\frac{\cos^2\theta}{1 - \tan^2\theta} + \frac{\sin^2\theta}{1 - \cot^2\theta} - 1 = 0$$

Solution

$$\begin{aligned} L.H.S. &= \frac{\cos^2\theta}{1 - \tan^2\theta} + \frac{\sin^2\theta}{1 - \cot^2\theta} - 1 \\ &= \frac{\cos^2\theta}{\sin^2\theta + \cos^2\theta - \tan^2\theta} + \frac{\sin^2\theta}{\sin^2\theta + \cos^2\theta - \cot^2\theta} - 1 \\ &= \frac{\cos^2\theta}{\sin^2\theta + \cos^2\theta - \frac{\sin^2\theta}{\cos^2\theta}} + \frac{\sin^2\theta}{\sin^2\theta + \cos^2\theta - \frac{\cos^2\theta}{\sin^2\theta}} - 1 \end{aligned}$$

$$\begin{aligned}
&= \frac{\cos^4\theta}{\cos^2\theta(\sin^2\theta+\cos^2\theta)-\sin^2\theta} \\
&\quad + \frac{\sin^4\theta}{\sin^2\theta(\sin^2\theta+\cos^2\theta)-\cos^2\theta} - 1 \\
&= \frac{\cos^4\theta}{\cos^2\theta-\sin^2\theta} + \frac{\sin^4\theta}{\sin^2\theta-\cos^2\theta} - 1 \\
&= \frac{\cos^4\theta}{\cos^2\theta-\sin^2\theta} - \frac{\sin^4\theta}{\cos^2\theta-\sin^2\theta} - 1 \qquad (8.2.1) \\
&= \frac{\cos^4\theta-\sin^4\theta}{\cos^2\theta-\sin^2\theta} - 1 \\
&= \frac{(\cos^2\theta-\sin^2\theta)(\cos^2\theta+\sin^2\theta)}{\cos^2\theta-\sin^2\theta} - 1 \\
&= \cos^2\theta+\sin^2\theta-1 \\
&= 1-1 \\
&= 0 = RHS
\end{aligned}$$

which completes the proof.

Example 8.2.4

Prove that,

$$\frac{\cos\theta}{1+\sin\theta} + \frac{1+\sin\theta}{\cos\theta} = 2\sec\theta$$

Solution

$$\begin{aligned}
\frac{\frac{\cos\theta}{\cos\theta}}{\frac{1+\sin\theta}{\cos\theta}} + \frac{\frac{1+\sin\theta}{\cos\theta}}{\frac{\cos\theta}{\cos\theta}} &= \frac{1}{\sec\theta+\tan\theta} + \frac{\sec\theta+\tan\theta}{1} \\
&= \frac{1+(\sec\theta+\tan\theta)^2}{\sec\theta+\tan\theta} \\
&= \frac{1+\sec^2\theta+2\sec\theta\tan\theta+\tan^2\theta}{\sec\theta+\tan\theta} \\
&= \frac{2\sec^2\theta+2\sec\theta\tan\theta}{\sec\theta+\tan\theta}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{2\sec^2\theta + 2\sec^2\theta\sin\theta}{\sec\theta(1+\sin\theta)} \\
&= \frac{2\sec^2\theta(1+\sin\theta)}{\sec\theta(1+\sin\theta)} \\
&= 2\sec\theta
\end{aligned}
$$

which completes the proof.

8.3 Compound Angles

We shall in this section, consider the identities for compound angles i.e., the sum or difference of angles $(A \pm B)$. The following identities:

(i) $\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$

(ii) $\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$

is true for any two angles A and B.
The proof is as follows:

In the following proof, $\angle A$ and $\angle B$ are both taken to be acute as well as $\angle(A+B)$ and $(\angle A > \angle B)$, but the results are valid for all angles of any magnitude.

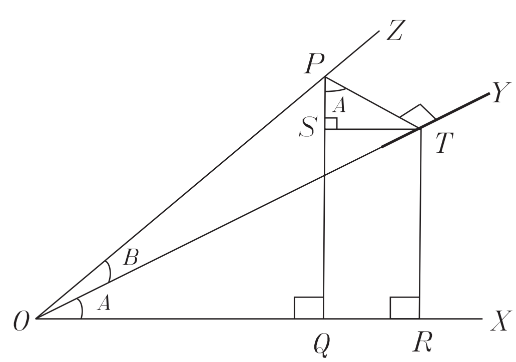

Figure 8.13: Compound Angle

From Figure 8.13, angle $XOY = \angle A$, $\angle YOZ = \angle B$, P is any point on OZ; PQ, PT are perpendiculars from P to OX and OY

respectively; while TR and TS are perpendiculars from T to OX and PQ respectively.

Since PQ and PT are perpendicular to OX and OY;

$$\therefore \quad \angle TPS = \angle A$$

From the construction, $SQRT$ is a rectangle, hence;

$$SQ = TR; \quad ST = QR$$

$$\begin{aligned}
\therefore \quad \sin(A+B) &= \frac{PQ}{OP} \\
&= \frac{PS+SQ}{OP} \\
&= \frac{PS}{OP} + \frac{SQ}{OP} \\
&= \frac{TR}{OP} + \frac{PS}{OP} \\
&= \frac{TR}{TO}.\frac{TO}{OP} + \frac{PS}{PT}.\frac{PT}{OP} \\
&= \sin A \cos B + \cos A \sin B
\end{aligned} \tag{8.3.1}$$

Similarly,

$$\begin{aligned}
\cos(A+B) &= \frac{OQ}{OP} \\
&= \frac{OR-QR}{OP} \\
&= \frac{OR}{OP} - \frac{QR}{OP} \\
&= \frac{OR}{OP} - \frac{ST}{OP} \\
&= \frac{OR}{TO}.\frac{TO}{OP} - \frac{ST}{PT}.\frac{PT}{OP} \\
&= \cos A \cos B - \sin A \sin B
\end{aligned} \tag{8.3.2}$$

Now, if we put $-B$ for B in (8.3.1) and (8.3.2) respectively, we obtain;

$$\begin{aligned}
\sin(A-B) &= \sin A \cos(-B) + \cos A \sin(-B) \\
&= \sin A \cos B - \cos A \sin B
\end{aligned} \tag{8.3.3}$$

Since $\cos(-B) = \cos B$ and $\sin(-B) = -\sin B$
Then,

$$\begin{aligned} \cos(A - B) &= \cos A \cos(-B) + \sin A \sin(-B) \\ &= \cos A \cos B + \sin A \sin B \end{aligned} \tag{8.3.4}$$

Putting the results of (8.3.1) to (8.3.4) together, we obtain the following identities:

$$\left.\begin{aligned} \sin(A \pm B) &= \sin A \cos B \pm \cos A \sin B \\ \cos(A \pm B) &= \cos A \cos B \mp \sin A \sin B \end{aligned}\right\} \tag{8.3.5}$$

Furthermore, from the identity, $\tan A = \frac{\sin A}{\cos A}$, we obtain;

$$\begin{aligned} \tan(A + B) &= \frac{\sin(A + B)}{\cos(A + B)} \\ &= \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B} \\ &= \frac{\frac{\sin A \cos B}{\cos A \cos B} + \frac{\cos A \sin B}{\cos A \cos B}}{\frac{\cos A \cos B}{\cos A \cos B} - \frac{\sin A \sin B}{\cos A \cos B}} \\ &= \frac{\frac{\sin A}{\cos A} + \frac{\sin B}{\cos B}}{1 - \frac{\sin A}{\cos A} . \frac{\sin B}{\cos B}} \\ &= \frac{\tan A + \tan B}{1 - \tan A \tan B} \end{aligned} \tag{8.3.6}$$

and,

$$\begin{aligned} \tan(A - B) &= \frac{\sin(A - B)}{\cos(A - B)} \\ &= \frac{\sin A \cos B - \cos A \sin B}{\cos A \cos B + \sin A \sin B} \\ &= \frac{\frac{\sin A \cos B}{\cos A \cos B} + \frac{\cos A \sin B}{\cos A \cos B}}{\frac{\cos A \cos B}{\cos A \cos B} - \frac{\sin A \sin B}{\cos A \cos B}} \\ &= \frac{\tan A - \tan B}{1 + \tan A \tan B} \end{aligned} \tag{8.3.7}$$

Alternatively, simply put $-B$ for B in (8.3.6) to get;

$$\begin{aligned} \tan(A - B) &= \frac{\tan A + \tan(-B)}{1 - \tan A \tan(-B)} \\ &= \frac{\tan A - \tan B}{1 + \tan A \tan B} \end{aligned}$$

These two results can formally be written as;

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B} \qquad (8.3.8)$$

Please note these identities, they are very important.

Example 8.3.1

Without using tables and calculator, find the values of $\sin 210°$, $\cos 150°$ and $\tan 15°$.

Solution

$$\begin{aligned} \sin 210° &= \sin(180° + 30°) \\ &= \sin 180° \cos 30° + \cos 180° \sin 30° \\ &= 0 + \left(-1.\frac{1}{2}\right) \\ &= -\frac{1}{2} \end{aligned}$$

$$\begin{aligned} \cos 150° &= \cos(90° + 60°) \\ &= \cos 90° \cos 60° - \sin 90° \sin 60° \\ &= 0 - \left(1.\frac{\sqrt{3}}{2}\right) \\ &= -\frac{\sqrt{3}}{2} \end{aligned}$$

$$\begin{aligned}
\tan 15^\circ &= \tan(45^\circ - 30^\circ) \\
&= \frac{\tan 45^\circ - \tan 30^\circ}{1 + \tan 45^\circ \tan 30^\circ} \\
&= \frac{1 - 1/\sqrt{3}}{1 + 1/\sqrt{3}} \\
&= \frac{\sqrt{3} - 1)(\sqrt{3} - 1)}{(\sqrt{3} + 1)(\sqrt{3} - 1)} \\
&= \frac{3 + 1 - 2\sqrt{3}}{3 - 1} \\
&= 2 - \sqrt{3}
\end{aligned}$$

Example 8.3.2

Find the values of $\sin(A + B)$, $\cos(A + B)$ and $\tan(A + B)$, given that, $\sin A = \frac{4}{5}$ and $\cos B = \frac{12}{13}$ for;
(i) A and B both acute
(ii) B is acute A is obtuse.

Solution

From the identity, $\sin^2 A + \cos^2 A = 1$, if $\sin A = \frac{4}{5}$;
Then,

$$\begin{aligned}
\cos A &= \sqrt{1 - \sin^2 A} \\
&= \sqrt{1 - \frac{16}{25}} \\
&= \pm\frac{3}{5}
\end{aligned}$$

Alternatively,

$$\cos A = \frac{3}{5};\ \tan A = \frac{4}{3}$$

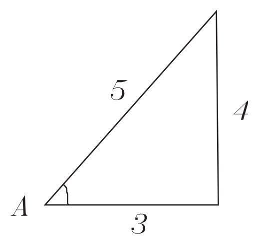

Figure 8.14:

Similarly, if $\cos B = \frac{12}{13}$, then;

$$\begin{aligned}\sin B &= \sqrt{1-\cos^2 B}\\ &= \sqrt{1-\frac{144}{169}}\\ &= \pm\frac{5}{13}\end{aligned}$$

Alternatively,

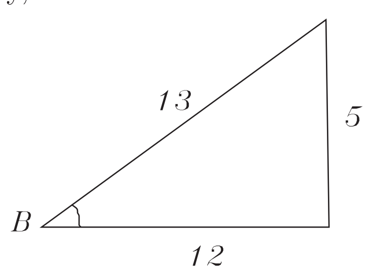

Figure 8.15:

$$\cos B = \frac{12}{13};\ \sin B = \frac{5}{13};\ \tan B = \frac{5}{12}$$

(i) For A and B acute, we have;

$$\begin{aligned}\sin(A+B) &= \sin A\cos B + \cos A\sin B\\ &= \frac{4}{5}.\frac{12}{13} + \frac{3}{5}.\frac{5}{13}\\ &= \frac{48}{65} + \frac{3}{13}\\ &= \frac{63}{65}\end{aligned}$$

$$\begin{aligned}\cos(A+B) &= \cos A\cos B - \sin A\sin B\\ &= \frac{3}{5}.\frac{12}{13} - \frac{4}{5}.\frac{5}{13}\\ &= \frac{36}{65} - \frac{4}{13}\\ &= \frac{16}{65}\end{aligned}$$

$$\begin{aligned}\tan(A+B) &= \frac{\sin(A+B)}{\cos(A+B)}\\ &= \frac{63/65}{16/65}\\ &= \frac{63}{16}\end{aligned}$$

(ii) For A obtuse, then;

$$\begin{aligned}\sin(A+B) &= \sin A\cos B + \cos A\sin B\\ &= \frac{4}{5}.\frac{12}{13} + \left(\frac{-3}{5}\right).\frac{5}{13}\\ &= \frac{33}{65}\end{aligned}$$

$$\begin{aligned}\cos(A+B) &= \cos A\cos B - \sin A\sin B\\ &= \left(\frac{-3}{5}\right).\frac{12}{13} - \frac{4}{5}.\frac{5}{13}\\ &= -\frac{56}{65}\end{aligned}$$

$$\begin{aligned}\tan(A+B) &= \frac{\sin(A+B)}{\cos(A+B)}\\ &= \frac{33}{65}.\left(-\frac{65}{56}\right)\\ &= -\frac{33}{56}\end{aligned}$$

Example 8.3.3

Simplify the following:

(i) $\cos A + \sin B \sin(A - B)$; (ii) $\dfrac{\tan(A-B) + \tan(A+B)}{1 - \tan(A-B)\tan(A+B)}$

Solution

(i)

$$\begin{aligned}
\cos A + \sin B \sin(A - B) &= \cos A \\
&\quad + \sin B(\sin A \cos B - \cos A \sin B) \\
&= \cos A + \sin A \sin B \cos B \\
&\quad - \cos A \sin^2 B \\
&= \cos A(1 - \sin^2 B) \\
&\quad + \sin A \sin B \cos B \\
&= \cos A \cos^2 B + \sin A \sin B \cos B \\
&= \cos B(\cos A \cos B + \sin A \sin B) \\
&= \cos B \cos(A - B)
\end{aligned}$$

(ii)

$$\begin{aligned}
\frac{\tan(A-B) + \tan(A+B)}{1 - \tan(A-B)\tan(A+B)} &= \tan[(A-B) + (A+B)] \\
&= \tan 2A
\end{aligned}$$

Remark 20 *Put $\theta = A - B$ and $\alpha = A + B$, then proceed by using the identity (8.3.8) to obtain your result.*

Example 8.3.4

(i) If $\tan\beta = \frac{7+4y}{1-3y}$ and $\tan\alpha = \frac{13+7y}{1-6y}$; show that $(\beta - \alpha)$ is independent of y.

(ii) Show that $\tan^{-1}(1/3) + \sin^{-1}(1/\sqrt{5}) = \pi/4$ without using tables or calculator.

(iii) Prove that $\cos\theta + \cos(\theta + 120°) + \cos(\theta + 240°) = 0$

Solution

(i) $\tan\beta = \frac{7+4y}{1-3y}$, $\tan\alpha = \frac{13+7y}{1-6y}$, thus;

$$\begin{aligned}
\tan(\beta-\alpha) &= \frac{\tan\beta - \tan\alpha}{1+\tan\beta\tan\alpha} \\
&= \frac{\frac{7+4y}{1-3y} - \frac{13+7y}{1-6y}}{1+\left(\frac{7+4y}{1-3y}\right)\left(\frac{13+7y}{1-6y}\right)} \\
&= \frac{(7+4y)(1-6y)-(13+7y)(1-3y)}{(1-3y)(1-6y)+(7+4y)(13+7y)} \\
&= \frac{7-38y-24y^2-13+32y+21y^2}{1-9y+18y^2+91+101y+28y^2} \\
&= \frac{-6-6y-3y^2}{92+92y+46y^2} \\
&= \frac{-3(2+2y+y^2)}{46(2+2y+y^2)} \\
&= \frac{-3}{46} \\
\therefore \quad \beta-\alpha &= \tan^{-1}\left(\frac{-3}{46}\right)
\end{aligned}$$

So, $(\beta-\alpha)$ is independent of y.

(ii) Let $\theta = \tan^{-1}\left(\frac{1}{3}\right)$, and $\alpha = \sin^{-1}\left(\frac{1}{\sqrt{5}}\right)$, then $\tan\theta = \frac{1}{3}$ and $\sin\alpha = \frac{1}{\sqrt{5}}$.
Drawing the appropriate right - angled triangles, we have;

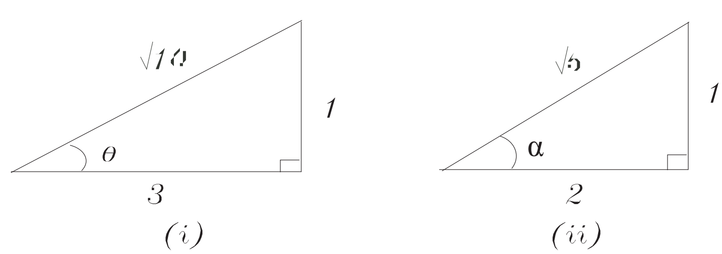

Figure 8.16:

Applying Pythagoras theorem, the 3^{rd} sides of the triangles becomes $\sqrt{10}$ and 2 units respectively. At this stage, we apply any of the identities for compound angles in Section 18.3, so that, using $\sin(\theta+\alpha)$ we have;

$$\sin(\theta+\alpha) = \sin\theta\cos\alpha + \cos\theta\sin\alpha$$

Thus, from triangles (i) and (ii) above, we have;

$$\sin\theta = \frac{1}{\sqrt{10}}; \quad \cos\theta = \frac{3}{\sqrt{10}}; \quad \sin\alpha = \frac{1}{\sqrt{5}}, \quad \cos\alpha = \frac{2}{\sqrt{5}}$$

So that,

$$\begin{aligned} \sin(\theta+\alpha) &= \frac{1}{\sqrt{10}}.\frac{2}{\sqrt{5}} + \frac{3}{\sqrt{10}}.\frac{1}{\sqrt{5}} \\ &= \frac{2}{5\sqrt{2}} + \frac{3}{5\sqrt{2}} \\ &= \frac{1}{\sqrt{2}} \\ \therefore \qquad \theta+\alpha &= \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) \\ &= 45^\circ \\ &= \frac{\pi}{4} \end{aligned}$$

Hence,

$$\begin{aligned} \theta+\alpha &= \tan^{-1}\left(\frac{1}{3}\right) + \sin^{-1}\left(\frac{1}{\sqrt{5}}\right) \\ &= \frac{\pi}{4} \end{aligned}$$

(iii) The equation can be re-written as;

$$\begin{aligned} \cos\theta \quad &+ \cos\theta\cos 120^\circ - \sin\theta\sin 120^\circ \\ &+ \cos\theta\cos 240^\circ - \sin\theta\sin 240^\circ = 0 \end{aligned}$$

$$\begin{aligned}
L.H.S. &= \cos\theta(1+\cos 120^\circ+\cos 240^\circ) \\
&\quad -\sin\theta(\sin 120^\circ+\sin 240^\circ) \\
&= \cos\theta(1-\cos 60^\circ-\cos 60^\circ) \\
&\quad -\sin\theta(\sin 60^\circ-\sin 60^\circ) \\
&= \cos\theta\left(1-\frac{1}{2}-\frac{1}{2}\right)-0 \\
&= 0 = R.H.S.
\end{aligned}$$

Thus,

$$\cos\theta+\cos(\theta+120^\circ)+\cos(\theta+240^\circ)=0$$

8.4 Multiple and Submultiple Angles

Now, for $B = A$ in each of these identities in Section 8.3, then;

$$\begin{aligned}
\sin(A+A) &= \sin A\cos A+\cos A\sin A \\
\text{i.e.}\quad \sin 2A &= 2\sin A\cos A
\end{aligned} \tag{8.4.1}$$

Similarly,

$$\begin{aligned}
\cos(A+A) &= \cos A\cos A-\sin A\sin A \\
\text{i.e.}\quad \cos 2A &= \cos^2 A-\sin^2 A \\
&= \cos^2 A-(1-\cos^2 A) \\
&= 2\cos^2 A-1 = 1-2\sin^2 A
\end{aligned} \tag{8.4.2}$$

and,

$$\begin{aligned}
\tan(A+A) &= \frac{\tan A+\tan A}{1-\tan A\tan A} \\
\tan 2A &= \frac{2\tan A}{1-\tan^2 A}
\end{aligned} \tag{8.4.3}$$

Observe that the last part of (8.4.2) can be re-written as;

$$\left.\begin{aligned}
\cos^2 A &= \tfrac{1}{2}(1+\cos 2A) \\
\sin^2 A &= \tfrac{1}{2}(1-\cos 2A)
\end{aligned}\right\} \tag{8.4.4}$$

Thus, if we replace the angle A of (8.4.1) to (8.4.3) with $\frac{1}{2}A$, the following result are obtained;

$$\sin A = 2\sin\frac{1}{2}A\cos\frac{1}{2}A \tag{8.4.5}$$

$$\cos A = \cos^2 \frac{1}{2}A - \sin^2 \frac{1}{2}A \tag{8.4.6}$$

$$= 2\cos^2 \frac{1}{2}A - 1 = 1 - 2\sin^2 \frac{1}{2}A \tag{8.4.7}$$

$$\tan A = \frac{2\tan \frac{1}{2}A}{1 - \tan^2 \frac{1}{2}A} \tag{8.4.8}$$

Note also that we can re-write (8.4.6)

$$\begin{aligned} \cos^2 \tfrac{1}{2}A &= \tfrac{1}{2}(1 + \cos A) \\ \sin^2 \tfrac{1}{2}A &= \tfrac{1}{2}(1 - \cos A) \end{aligned} \tag{8.4.9}$$

Furthermore, if we extend our earlier multiple angle in (8.4.1) - (8.4.3) by replacing the angle B in (8.3.5) and (8.3.8) with $2A$ in each of the identities then

$$\begin{aligned} \sin(A + 2A) &= \sin A \cos 2A + \cos A \sin 2A \\ \sin 3A &= \sin A(1 - 2\sin^2 A) + \cos A(2\sin A \cos A) \\ &= \sin A - 2\sin^3 A + 2\sin A \cos^2 A \\ &= \sin A - 2\sin^3 A + 2\sin A(1 - \sin^2 A) \\ &= \sin A - 2\sin^3 A + 2\sin A - 2\sin^3 A \\ &= 3\sin A - 4\sin^3 A \end{aligned} \tag{8.4.10}$$

$$\begin{aligned} \cos(A + 2A) &= \cos A \cos 2A - \sin A \sin 2A \\ \cos 3A &= \cos A(2\cos^2 A - 1) - \sin A(2\sin A \cos A) \\ &= 2\cos^3 A - \cos A - 2\sin^2 A \cos A \\ &= 2\cos^3 A - \cos A - 2(1 - \cos^2 A)\cos A \\ &= 2\cos^3 A - \cos A - 2\cos A + 2\cos^3 A \\ &= 4\cos^3 - 3\cos A \end{aligned} \tag{8.4.11}$$

$$\begin{aligned}
\tan(A+2A) &= \frac{\tan A + \tan 2A}{1 - \tan A \tan 2A} \\
\tan 3A &= \frac{\tan A + \frac{2\tan A}{1-\tan^2 A}}{1 - \tan A(\frac{2\tan A}{1-\tan^2 A})} \\
&= \frac{\frac{\tan A(1-\tan^2 A)+2\tan A}{1-\tan^2 A}}{\frac{1-\tan^2 A-\tan A(2\tan A)}{1-\tan^2 A}} \\
&= \frac{\tan A(1-\tan^2 A)+2\tan A}{1-\tan^2 A-2\tan^2 A} \\
&= \frac{\tan A - \tan^3 A + 2\tan A}{1-3\tan^2 A} \\
&= \frac{3\tan A - \tan^3 A}{1-3\tan^2 A}
\end{aligned} \tag{8.4.12}$$

Example 8.4.1

(i) Without using tables or calculator, find the sine, cosine and tangent of $67\frac{1}{2}^\circ$

(ii) If $t = \tan\frac{1}{2}\theta$, find expressions for $\sin\theta, \cos\theta$ and $\tan\theta$ in terms of t. Hence, express the square root of:

$$\frac{1+\sin\theta}{3\sin\theta + 4\cos\theta + 5}$$

in terms of t and find the angle between 0° and 360° that satisfies the equation homogeneously.

Solution

(i) Using the formula of (8.4.4), we have;

$$\begin{aligned}
\sin^2 67\frac{1}{2}^\circ &= \frac{1}{2}\left[1-\cos 2\left(67\frac{1}{2}^\circ\right)\right]\\
&= \frac{1}{2}(1-\cos 135^\circ)\\
&= \frac{1}{2}(1+\cos 45^\circ)\\
&= \frac{1}{2}\left(1+\frac{1}{\sqrt{2}}\right)\\
&= \frac{1}{2}\left(1+\frac{\sqrt{2}}{2}\right)\\
&= \frac{1}{4}(2+\sqrt{2})\\
\therefore \qquad \sin 67\frac{1}{2}^\circ &= \frac{1}{2}\sqrt{(2+\sqrt{2})}
\end{aligned}$$

Similarly,

$$\begin{aligned}
\cos^2 67\frac{1}{2}^\circ &= \frac{1}{2}\left[1+\cos 2(67\frac{1}{2}^\circ)\right]\\
&= \frac{1}{2}(1+\cos 135^\circ)\\
&= \frac{1}{2}(1-\cos 45^\circ)\\
&= \frac{1}{2}(1-\frac{1}{\sqrt{2}})\\
&= \frac{1}{2}(1-\frac{\sqrt{2}}{2})\\
&= \frac{1}{4}(2-\sqrt{2})\\
\therefore \qquad \cos 67\frac{1}{2}^\circ &= \frac{1}{2}\sqrt{(2-\sqrt{2})}
\end{aligned}$$

$$\begin{aligned}
\tan 67\frac{1}{2}^\circ &= \frac{\sin 67\frac{1}{2}^\circ}{\cos 67\frac{1}{2}^\circ} \\
&= \frac{\frac{1}{2}\surd(2+\sqrt{2})}{\frac{1}{2}\surd(2-\sqrt{2})} \\
&= \surd\left(\frac{2+\sqrt{2}}{2-\sqrt{2}}\right) \\
&= \surd\left(\frac{(2+\sqrt{2})(2-\sqrt{2})}{(2-\sqrt{2})^2}\right) \\
&= \frac{\sqrt{2}}{2-\sqrt{2}} \\
&= \frac{2(\surd 2+1)}{2} \\
&= \surd 2+1
\end{aligned}$$

(ii) If $t = \tan\frac{\theta}{2}$, then,

$$\begin{aligned}
\sin\theta &= \sin\left(\frac{\theta}{2}+\frac{\theta}{2}\right) \\
&= 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} \\
&= \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{\sin^2\frac{\theta}{2}+\cos^2\frac{\theta}{2}}; \ \left(\text{since } \sin^2\frac{\theta}{2}+\cos^2\frac{\theta}{2}=1\right)
\end{aligned}$$

i.e.,

$$\begin{aligned}
\frac{2\left(\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}\right)}{\left(\frac{\sin^2\frac{\theta}{2}}{\cos^2\frac{\theta}{2}}\right)+1} &= \frac{2\tan\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}} \\
&= \frac{2t}{1+t^2}
\end{aligned}$$

$$\therefore \quad \sin\theta = \frac{2t}{1+t^2}$$

$$\begin{aligned}
\cos\theta &= \cos\left(\frac{\theta}{2}+\frac{\theta}{2}\right)\\
&= \cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2}\\
&= \frac{\cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2}}{\cos^2\frac{\theta}{2}+\sin^2\frac{\theta}{2}};\ (\text{Since }\ \cos^2\frac{\theta}{2}+\sin^2\frac{\theta}{2}=1)\\
&= \frac{1-\frac{\sin^2\frac{\theta}{2}}{\cos^2\frac{\theta}{2}}}{1+\frac{\sin^2\frac{\theta}{2}}{\cos^2\frac{\theta}{2}}}\\
&= \frac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}\\
&= \frac{1-t^2}{1+t^2}\\
\therefore \qquad \cos\theta &= \frac{1-t^2}{1+t^2}
\end{aligned}$$

$$\begin{aligned}
\tan\theta &= \frac{\sin\theta}{\cos\theta}\\
&= \frac{\frac{2t}{1+t^2}}{\frac{1-t^2}{1+t^2}}\\
&= \frac{2t}{1-t^2}
\end{aligned}$$

Using the above results,

$$\frac{1+\sin\theta}{3\sin\theta+4\cos\theta+5}$$

reduces to,

$$\begin{aligned}
\frac{1+\frac{2t}{1+t^2}}{3\left(\frac{2t}{1+t^2}\right)+4\left(\frac{1-t^2}{1+t^2}\right)+5} &= \frac{\frac{1+t^2+2t}{1+t^2}}{\frac{6t+4(1-t^2)+5(1+t^2)}{1+t^2}} \\
&= \frac{1+t^2+2t}{6t+4(1-t^2)+5(1+t^2)} \\
&= \frac{(1+t)^2}{t^2+6t+9} \\
&= \frac{(1+t)^2}{(t+3)^2} \\
&= \left(\frac{1+t}{t+3}\right)^2
\end{aligned}$$

Thus, we have, $\pm\left(\frac{1+t}{t+3}\right)$ as the required square root in terms of t.

Now, to find the value of t, we have,

$$\begin{aligned}
\left(\frac{1+t}{t+3}\right)^2 &= 0 \\
\text{or} \quad t &= -1 \\
\therefore \quad \frac{\theta}{2} &= \tan^{-1}(-1) \\
&= n\pi - 45^\circ; \quad \pm n = 0, 1, 2, \ldots
\end{aligned}$$

So that,

$$\theta = 2n\pi - 90^\circ; \quad \pm n = 0, 1, 2, \ldots$$

Hence, in $[0^\circ, 360^\circ]$, we have,

$$\begin{aligned}
\theta &= 2n180^\circ - 90^\circ \\
&= 360^\circ - 90^\circ \\
&= 270^\circ
\end{aligned}$$

Example 8.4.2

(i) Without using tables or calculator, prove that for all values of the angle θ,

$$\sin^2\theta + \sin^2(\theta + 120^\circ) + \sin^2(\theta - 120^\circ) = \frac{3}{2}$$

(ii) Find the general value of θ in degrees, which satisfies simultaneously the equations $\tan\theta = \sqrt{3}$; $\sec\theta = -2$

(iii) Prove that, $4\cos\theta\cos(\theta + \frac{2\pi}{3})\cos(\theta + \frac{4\pi}{3}) = \cos 3\theta$

Solution

(i)

$$\begin{aligned}
L.H.S. &= \sin^2\theta + (\sin\theta\cos 120^\circ + \cos\theta\sin 120^\circ)^2 \\
&\quad +(\sin\theta\cos 120^\circ - \cos\theta\sin 120^\circ)^2 \\
&= \sin^2\theta + (-\sin\theta\cos 60^\circ + \cos\theta\sin 60^\circ)^2 \\
&\quad +(-\sin\theta\cos 60^\circ - \cos\theta\sin 60^\circ)^2 \\
&= \sin^2\theta + \left(\frac{\sqrt{3}}{2}\cos\theta - \frac{1}{2}\sin\theta\right)^2 \\
&\quad + \left(-\frac{1}{2}\sin\theta - \frac{\sqrt{3}}{2}\cos\theta\right)^2 \\
&= \sin^2\theta + \frac{3}{4}\cos^2\theta - \frac{\sqrt{3}}{2}\cos\theta\sin\theta + \frac{1}{4}\sin^2\theta \\
&\quad +\frac{1}{4}\sin^2\theta + \frac{\sqrt{3}}{2}\sin\theta\cos\theta + \frac{3}{4}\cos^2\theta \\
&= \sin^2\theta + \frac{1}{2}\sin^2\theta + \frac{3}{2}\cos^2\theta \\
&= \frac{3}{2}\sin^2\theta + \frac{3}{2}\cos^2\theta \\
&= \frac{3}{2}(\sin^2\theta + \cos^2\theta) \\
&= \frac{3}{2} = R.H.S.
\end{aligned}$$

$$\therefore \quad \sin^2\theta + \sin^2(\theta + 120^\circ) + \sin^2(\theta - 120^\circ) = \frac{3}{2}$$

(ii)

$$\tan\theta = \sqrt{3} \qquad (i)$$

$$\sec\theta = -2 \qquad (ii)$$

From (i),

$$\frac{\sin\theta}{\cos\theta} = \sqrt{3} \ \text{ or } \ \sin\theta = \sqrt{3}\cos\theta$$

Thus,

$$\sin\theta - \sqrt{3}\cos\theta = 0 \qquad (iii)$$

But from (ii),

$$\frac{1}{\cos\theta} = -2 \ \text{ or } \ \cos\theta = -\frac{1}{2}$$

Substituting for $\cos\theta$ in (iii), we have,

$$\begin{aligned}
\sin\theta - \sqrt{3}(-\frac{1}{2}) &= 0 \\
\sin\theta + \frac{\sqrt{3}}{2} &= 0 \\
\sin\theta &= -\frac{\sqrt{3}}{2} \\
\therefore \qquad \theta &= \sin^{-1}(-\frac{\sqrt{3}}{2}) \\
&= n\pi + (180^\circ - (-60^\circ)); \ n = 0, 2, 4, \ldots \\
&= n\pi + 240^\circ; \ n = 0, 2, 4,
\end{aligned}$$

(iii) Re-writing, we have,

$$\begin{aligned}
L.H.S &= 4\cos\theta\cos(\theta+120^\circ)\cos(\theta+240^\circ) \\
&= 4\cos\theta(\cos\theta\cos 120^\circ - \sin\theta\sin 120^\circ) \\
&\quad \times(\cos\theta\cos 240^\circ - \sin\theta\sin 240^\circ) \\
&= 4\cos\theta(-\cos\theta\cos 60^\circ - \sin\theta\sin 60^\circ) \\
&\quad \times(-\cos\theta\cos 60^\circ + \sin\theta\sin 60^\circ) \\
&= 4\cos\theta(-\frac{1}{2}\cos\theta - \frac{\sqrt{3}}{2}\sin\theta)(-\frac{1}{2}\cos\theta + \frac{\sqrt{3}}{2}\sin\theta) \\
&= 4\cos\theta(\frac{1}{4}\cos^2\theta - \frac{\sqrt{3}}{4}\sin\theta\cos\theta + \frac{\sqrt{3}}{4}\sin\theta\cos\theta \\
&\quad -\frac{3}{4}\sin^2\theta) \\
&= 4\cos\theta(\frac{1}{4}\cos^2\theta - \frac{3}{4}\sin^2\theta) \\
&= 4\cos\theta(\frac{1}{4}\cos^2\theta - \frac{3}{4}(1-\cos^2\theta)) \\
&= 4\cos\theta(\cos^2\theta - \frac{3}{4}) \\
&= 4\cos^3\theta - 3\cos\theta \\
&= \cos 3\theta = R.H.S.
\end{aligned}$$

Example 8.4.3

Prove that,

(i) $\cos^4\theta - \sin^4\theta = \cos 2\theta$ and that, $\cos^4 + \sin^4\theta = 1 - \frac{1}{2}\sin^2 2\theta$

(ii) $\sin 2\theta = \frac{2\tan\theta}{1+\tan^2\theta}$

(iii) $\cos 2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta}$

Solution

(i) To show that $\cos^4\theta - \sin^4\theta = \cos 2\theta$, we have,

$$\begin{aligned}
\cos^4\theta - \sin^4\theta &= (\cos^2\theta + \sin^2\theta)(\cos^2\theta - \sin^2\theta) \\
&= \cos^2\theta - \sin^2\theta \\
&= \cos 2\theta
\end{aligned}$$

Hence,

$$\begin{aligned}\cos^4\theta+\sin^4\theta &= (\cos^2\theta+\sin^2\theta)^2-2\sin^2\theta\cos^2\theta\\ &= 1-2\sin^2\theta\cos^2\theta\\ &= 1-\frac{1}{2}(2\sin\theta\cos\theta)^2\\ &= 1-\frac{1}{2}(\sin 2\theta)^2\\ &= 1-\frac{1}{2}\sin^2 2\theta\end{aligned}$$

(ii) We want to show that $\sin 2\theta=\frac{2\tan\theta}{1+\tan^2\theta}$, thus

$$\begin{aligned}\sin 2\theta &= \frac{2\sin\theta\cos\theta}{1}\\ &= \frac{\frac{2\sin\theta\cos\theta}{\cos^2\theta}}{\frac{1}{\cos^2\theta}}\\ &= \frac{2\tan\theta}{\sec^2\theta}\\ &= \frac{2\tan\theta}{1+\tan^2\theta}\end{aligned}$$

(iii) We need to show that $\cos 2\theta=\frac{1-\tan^2\theta}{1+\tan^2\theta}$, thus

$$\begin{aligned}\cos 2\theta &= \frac{\cos^2\theta-\sin^2\theta}{1}\\ &= \frac{\cos^2\theta-\sin^2\theta}{\cos^2\theta+\sin^2\theta};\ (\cos^2\theta+\sin^2\theta=1)\\ &= \frac{\frac{\cos^2\theta}{\cos^2\theta}-\frac{\sin^2\theta}{\cos^2\theta}}{\frac{\cos^2\theta}{\cos^2\theta}+\frac{\sin^2\theta}{\cos^2\theta}}\\ &= \frac{1-\tan^2\theta}{1+\tan^2\theta}\end{aligned}$$

8.5 The Factor Formulae

These formulae deals with the sum and difference of two sines and of two cosines. We could recall the identities (8.3.5);

$$\begin{aligned}\sin(A+B) &= \sin A\cos B+\cos A\sin B\\ \sin(A-B) &= \sin A\cos B-\cos A\sin B\end{aligned}$$

adding the two identities, we obtain,

$$\sin(A+B)+\sin(A-B)=2\sin A\cos B \tag{8.5.1}$$

Subtracting second from first, we obtain,

$$\sin(A+B)-\sin(A-B)=2\cos A\sin B \tag{8.5.2}$$

Similarly, from the identities (8.3.5)

$$\begin{aligned}\cos(A+B) &= \cos A\cos B-\sin A\sin B\\ \cos(A-B) &= \cos A\cos B+\sin A\sin B\end{aligned}$$

On addition of the two identities, we have,

$$\cos(A+B)+\cos(A-B)=2\cos A\cos B \tag{8.5.3}$$

Subtracting second from first gives,

$$\cos(A-B)-\cos(A+B)=2\sin A\sin B \tag{8.5.4}$$

Now, let $A+B=x$ and $A-B=y$, so that,

$$\begin{aligned}2A &= x+y \text{ or } A=\frac{1}{2}(x+y)\\ 2B &= x-y \text{ or } B=\frac{1}{2}(x-y)\end{aligned}$$

Thus, the results of (8.5.1) to (8.5.4) become,

$$\left.\begin{aligned}\sin x+\sin y &= 2\sin\left(\tfrac{x+y}{2}\right)\cos\left(\tfrac{x-y}{2}\right)\\ \sin x-\sin y &= 2\cos\left(\tfrac{x+y}{2}\right)\sin\left(\tfrac{x-y}{2}\right)\\ \cos x+\cos y &= 2\cos\left(\tfrac{x+y}{2}\right)\cos\left(\tfrac{x-y}{2}\right)\\ \cos y-\cos x &= 2\sin\left(\tfrac{x+y}{2}\right)\sin\left(\tfrac{x-y}{2}\right)\end{aligned}\right\} \tag{8.5.5}$$

Example 8.5.1

Find the product form of the following:

(i) $\sin 4\theta - \sin 2\theta$

(ii) $\cos\varphi + \cos 3\varphi$

(iii) $\sin 5\theta + \sin 3\theta$

Solution

Using our factor formulae (8.5.5);

(i) The product form of $\sin 4\theta - \sin 2\theta$ is given as;

$$\begin{aligned}\sin 4\theta - \sin 2\theta &= 2\cos\left(\frac{4\theta + 2\theta}{2}\right)\sin\left(\frac{4\theta - 2\theta}{2}\right)\\ &= 2\cos 3\theta \sin\theta\end{aligned}$$

(ii) We obtain the product form of $\cos\varphi + \cos 3\varphi$ thus,

$$\begin{aligned}\cos\varphi + \cos 3\varphi &= 2\cos\left(\frac{\varphi + 3\varphi}{2}\right)\cos\left(\frac{\varphi - 3\varphi}{2}\right)\\ &= 2\cos 2\varphi\cos(-\varphi)\\ &= 2\cos 2\varphi\cos\varphi\end{aligned}$$

(iii) The product form of $\sin 5\theta + \sin 3\theta$ gives;

$$\begin{aligned}\sin 5\theta + \sin 3\theta &= 2\sin\left(\frac{5\theta + 3\theta}{2}\right)\cos\left(\frac{5\theta - 3\theta}{2}\right)\\ &= 2\sin 4\theta\cos\theta\end{aligned}$$

Example 8.5.2

Express the following products as the sum of two sines or two cosines:

(i) $2\sin\theta\sin 2\theta$

(ii) $2\sin 3\theta\cos 6\theta$

(iii) $\cos(n\alpha + \theta)\cos(n\alpha - \theta)$

Solution

(i) To express $2\sin\theta\sin 2\theta$ as the sum of two cosines, we have,

$$\begin{aligned}2\sin\theta\sin 2\theta &= \cos(\theta-2\theta)-\cos(\theta+2\theta)\\ &= \cos(-\theta)-\cos 3\theta\\ &= \cos\theta-\cos 3\theta\end{aligned}$$

(ii) Expressing $2\sin 3\theta\cos 6\theta$ as the sum of two sines gives;

$$\begin{aligned}2\sin 3\theta\cos 6\theta &= \sin(3\theta+6\theta)+\sin(3\theta-6\theta)\\ &= \sin 9\theta+\sin(-3\theta)\\ &= \sin 9\theta-\sin 3\theta\end{aligned}$$

(iii) Expressing $\cos(n\alpha+\theta)\cos(n\alpha-\theta)$ as the sum of two cosines gives;

$$\begin{aligned}\cos(n\alpha+\theta)\cos(n\alpha-\theta) &= \frac{1}{2}\{\cos[(n\alpha+\theta)+(n\alpha-\theta)]\\ &\quad +\cos[(n\alpha+\theta)-(n\alpha-\theta)]\}\\ &= \frac{1}{2}[\cos(2n\alpha)+\cos(2\theta)]\\ &= \frac{1}{2}[\cos(2n\alpha)+\cos 2\theta]\end{aligned}$$

Example 8.5.3

Show that,

(i) $\cos 5\theta+\cos 3\theta = 2\cos\theta\cos 4\theta$

(ii) $\cos 15^\circ - \cos 75^\circ = \frac{1}{\sqrt{2}}$

(iii) $\frac{\sin 3\theta\sin 6\theta+\sin\theta\sin 2\theta}{\sin 3\theta\cos 6\theta+\sin\theta\cos 2\theta} = \tan 5\theta$

(iv) $\frac{\cos 2\theta+\cos 5\theta+\cos 8\theta}{\sin 2\theta+\sin 5\theta+\sin 8\theta} = \cot 5\theta$

Solution

(i) To show that $\cos 5\theta + \cos 3\theta = 2\cos\theta\cos 4\theta$:

$$\begin{aligned}\cos 5\theta + \cos 3\theta &= 2\cos\left(\frac{5\theta+3\theta}{2}\right)\cos\left(\frac{5\theta-3\theta}{2}\right)\\ &= 2\cos 4\theta\cos\theta\\ &= 2\cos\theta\cos 4\theta = R.H.S.\end{aligned}$$

(ii) We want to establish that $\cos 15^\circ - \cos 75^\circ = \frac{1}{\sqrt{2}}$, thus

$$\begin{aligned}\cos 15^\circ - \cos 75^\circ &= 2\sin\left(\frac{75^\circ+15^\circ}{2}\right)\sin\left(\frac{75^\circ-15^\circ}{2}\right)\\ &= 2\sin 45^\circ\sin(30^\circ)\\ &= 2.\frac{1}{\sqrt{2}}.\frac{1}{2}\\ &= \frac{1}{\sqrt{2}}\end{aligned}$$

(iii) We need to show that $\frac{\sin 3\theta\sin 6\theta+\sin\theta\sin 2\theta}{\sin 3\theta\cos 6\theta+\sin\theta\cos 2\theta} = \tan 5\theta$ holds, thus

$$\begin{aligned}L.H.S. &= \frac{\frac{1}{2}\{\cos 9\theta - \cos(-3\theta) + \cos 3\theta - \cos(-\theta)\}}{\frac{1}{2}\{\sin 9\theta + \sin(-3\theta) + \sin 3\theta + \sin(-\theta)\}}\\ &= \frac{\cos 9\theta - \cos 3\theta + \cos 3\theta - \cos\theta}{\sin 9\theta - \sin 3\theta + \sin 3\theta - \sin\theta}\\ &= \frac{\cos 9\theta - \cos\theta}{\sin 9\theta - \sin\theta}\\ &= \frac{2\sin(\frac{9\theta+\theta}{2})\sin(\frac{9\theta-\theta}{2})}{2\cos(\frac{9\theta+\theta}{2})\sin(\frac{9\theta-\theta}{2})}\\ &= \frac{\sin 5\theta\sin 4\theta}{\cos 5\theta\sin 4\theta}\\ &= \tan 5\theta = R.H.S.\end{aligned}$$

(iv) To show that $\frac{\cos 2\theta+\cos 5\theta+\cos 8\theta}{\sin 2\theta+\sin 5\theta+\sin 8\theta} = \cot 5\theta$ is valid, we have

$$\begin{aligned}
L.H.S. &= \frac{\cos(5\theta - 3\theta) + \cos 5\theta + \cos(5\theta + 3\theta)}{\sin(5\theta - 3\theta) + \sin 5\theta + \sin(5\theta + 3\theta)} \\
&= \frac{\cos 5\theta \cos 3\theta + \sin 5\theta \sin 3\theta + \cos 5\theta + \cos 5\theta \cos 3\theta - \sin 5\theta \sin 3\theta}{\sin 5\theta \cos 3\theta - \cos 5\theta \sin 3\theta + \sin 5\theta + \sin 5\theta \cos 3\theta + \cos 5\theta \sin 3\theta} \\
&= \frac{2\cos 5\theta \cos 3\theta + \cos 5\theta}{2\sin 5\theta \cos 3\theta + \sin 5\theta} \\
&= \frac{\frac{2\cos 5\theta \cos 3\theta}{\sin 5\theta} + \frac{\cos 5\theta}{\sin 5\theta}}{\frac{2\sin 5\theta \cos 3\theta}{\sin 5\theta} + \frac{\sin 5\theta}{\sin 5\theta}}
\end{aligned}$$

$$\begin{aligned}
&= \frac{2\cot 5\theta \cos 3\theta + \cot 5\theta}{2\cos 3\theta + 1} \\
&= \frac{\cot 5\theta(2\cos 3\theta + 1)}{2\cos 3\theta + 1} \\
&= \cot 5\theta = R.H.S.
\end{aligned}$$

Example 8.5.4

(i) If $\theta = 18°$, prove that $\cos 3\theta = \sin 2\theta$, and hence, show that $\cos\theta$ is a root of the equation:
$4\cos^3\theta - 3\cos\theta - 2\sin\theta\cos\theta = 0$
Thus, use the result to find $\sin 18°$ without using tables or calculator.

(ii) Find the values of x in the range of $0°$ to $360°$ that satisfies each of the trigonometric equations:

(a) $\tan 2x \tan 4x = 1$

(b) $\cos 4x + 3\cos 2x + 2 = 0$

Solution

(i) If $\theta = 18°$, then,

$$\begin{aligned}
\cos 3\theta = \cos 54° &= \sin(90° - 54°) \\
&= \sin 36° \\
&= \sin 2\theta
\end{aligned}$$

But,

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$$

and,

$$\sin 2\theta = 2\sin\theta\cos\theta$$

So that, since $\cos 3\theta = \sin 2\theta$ then we have,

$$\begin{aligned} \cos 3\theta - \sin 2\theta &= 0 \\ \text{i.e.,}\ \ 4\cos^3\theta - 3\cos\theta - 2\sin\theta\cos\theta &= 0 \qquad (i) \end{aligned}$$

On factorizing (i), we have,

$$\begin{aligned} \cos\theta(4\cos^2\theta - 3 - 2\sin\theta) &= 0 \\ \cos\theta(4\cos^2\theta - 2\sin\theta - 3) &= 0 \\ \text{i.e.,}\ \ \cos\theta = 0 \ \text{ or } \ 4\cos^2\theta - 2\sin\theta - 3 &= 0 \qquad (ii) \end{aligned}$$

Clearly, $\cos\theta = 0$ or $\theta = 90^\circ$ is a root of the equation (i), but since $\theta = 18^\circ \neq 90^\circ$, therefore, θ satisfies (ii).

Now, solving (ii) quadratically, we have,

$$\begin{aligned} 4(1 - \sin^2\theta) - 2\sin\theta - 3 &= 0 \\ 4 - 4\sin^2\theta - 2\sin\theta - 3 &= 0 \\ 4\sin^2\theta + 2\sin\theta - 1 &= 0 \\ \sin\theta &= \frac{-2 \pm \sqrt{4+16}}{8} \\ &= \frac{-1 \pm \sqrt{5}}{4} \end{aligned}$$

But, $\sin 18^\circ$ is not negative, therefore only the positive sign can be used, such that,

$$\sin 18^\circ = \frac{\sqrt{5} - 1}{4}$$

(ii) (a) The values of $x : 0 \leq x \leq 360°$ that satisfies $\tan 2x \tan 4x = 1$ is given thus,

$$\begin{aligned}
\tan 2x \tan 4x &= 1 \\
\tan 2x \left(\frac{2\tan 2x}{1-\tan^2 2x}\right) &= 1 \\
2\tan^2 2x &= 1 - \tan^2 2x \\
3\tan^2 2x &= 1 \\
\tan^2 2x &= \frac{1}{3} \\
\tan 2x &= \pm\frac{1}{\sqrt{3}} \\
\therefore \quad 2x &= n\pi \pm 30°;\ n = 1, 2, 3, \ldots \\
x &= \frac{n\pi}{2} \pm 15°;\ n = 1, 2, \ldots \\
\therefore \quad x &= 105°, 75°, 165°, 195°
\end{aligned}$$

(b) The values of $x : 0 \leq x \leq 360°$ that satisfies $\cos 4x + 3\cos 2x + 2 = 0$ is given thus,

$$\begin{aligned}
\cos 4x + 3\cos 2x + 2 &= 0 \\
(\cos^2 2x - \sin^2 2x) + 3\cos 2x + 2 &= 0 \\
(\cos^2 2x - (1 - \cos^2 2x)) + 3\cos 2x + 2 &= 0 \\
2\cos^2 2x + 3\cos 2x + 1 &= 0 \\
(2\cos 2x + 1)(\cos 2x + 1) &= 0 \\
\cos 2x &= -\frac{1}{2} \quad \text{or} \quad -1
\end{aligned}$$

If $\cos 2x = -\frac{1}{2}$, then,

$$\begin{aligned}
2x &= 120° \\
\text{or} \quad x &= 60°
\end{aligned}$$

Similarly, if $\cos 2x = -1$, then,

$$\begin{aligned}
2x &= 180° \\
x &= 90° \\
\therefore \quad x &= 60 \text{ or } 90°
\end{aligned}$$

Chapter 9

Theory of Complex Numbers

In chapter 2, we observed that the type of root/solution obtained in a quadratic equation when the discriminant, D is zero (i.e. $D^2 = b^2 - 4ac = 0$) is a complex root. These sorts of solutions were not solvable in Chapter 2 because the set of real numbers is insufficient for solving such algebraic equations with complex roots. It is therefore the quest for the solution of this type of equations that gave rise to the subject of this chapter – Theory of Complex Numbers.

It follows from the axioms governing the relation, less than '$<$' and greater than '$>$' that the square of real number is never negative. For example, the elementary quadratic equation with negative discriminant (i.e. $x^2 = -1$) has no solution among the set of real numbers. To overcome this difficulty, we introduce new set of numbers called *complex numbers* which include the set of real numbers as a subset to provide solutions to such equations. We denote this new number by the symbol $\underline{i}$ and call it the imaginary unit. Thus, for the number $\underline{i}$ the equation $i^2 + 1 = 0$ (or $i^2 = -1$) holds true. Furthermore, the set of real numbers are supplemented with new numbers which are called pure *imaginary numbers*, and are regarded as products of real numbers $\underline{b}$ and the imaginary unit $\underline{i}$. In addition, we write the sum of a real number $\underline{a}$ and an imaginary number $\underline{ib}$ in the form $a + ib$.

Definition 13 *By complex number, we mean an ordered pair of* *real numbers which we denote by $Z = (a, b)$ and are defined as expression of the form*

$$Z = a + ib$$

where $i^2 = -1$, the number $\underline{a}$ is called the real part of Z written ReZ, and $\underline{b}$ is called the imaginary part of Z, written ImZ. For example, $Z = 5 + 8i; ReZ = 5$ and $ImZ = 8$.

9.1 Algebra of Complex Numbers

(i) **Equality of Complex Numbers:** Two complex numbers $Z_1 = a + ib$ and $Z_2 = c + id$ are equal if and only if $a = c$ and $b = d$, i.e. if the real parts of Z are equal and the imaginary parts of Z are equal.

(ii) **Addition and Subtraction:** The sum or difference of two complex numbers gives another complex number. Suppose, $Z = a + ib,\ Z_2 = c + id$ then,

$$\begin{aligned} Z_1 + Z_2 &= (a + ib) + (c + id) \\ &= (a + c) + i(b + d) \end{aligned}$$

and

$$\begin{aligned} Z_1 - Z_2 &= (a + ib) - (c + id) \\ &= (a - c) + i(b - d) \end{aligned}$$

Example 9.1.1

Find the sum and difference of $Z_1 = 2 + 3i$ and $Z_2 = 7 - 13i$.

Solution

$$\begin{aligned} Z_1 + Z_2 &= (2 + 3i) + (7 - 13i) \\ &= (2 + 7) + i(3 - 13) \\ &= 9 - 10i \end{aligned}$$

and

$$\begin{aligned} Z_1 - Z_2 &= (2+3i) - (7-13i) \\ &= (2-7) + i(3+13) \\ &= -5+16i \end{aligned}$$

(iii) **Multiplication and Division:** Let $Z_1 = a + ib$ and $Z_2 = c + id$. Then we define the product $Z_1 Z_2$ as,

$$\begin{aligned} Z_1 Z_2 &= (a+ib)(c+id) \\ &= ac + iad + ibc + i^2 bd \\ &= ac + ibc + iad - bd; \ (\text{since } i^2 = -1) \\ &= ac - bd + i(bc + ad) \end{aligned}$$

Similarly, we define the quotient $\frac{Z_1}{Z_2}$ as,

$$\begin{aligned} \frac{Z_1}{Z_2} &= \frac{a+ib}{c+id} \\ &= \frac{a+ib}{c+id} \times \frac{c-id}{c-id} \\ &= \frac{ac - iad + ibc - i^2 bd}{c^2 - icd + icd - i^2 d^2} \\ &= \frac{ac + ibc - iad + bd}{c^2 + d^2} \\ &= \frac{(ac+bd) + i(bc-ad)}{c^2+d^2} \\ &= \frac{ac+bd}{c^2+d^2} + i\left(\frac{bc-ad}{c^2+d^2}\right) \end{aligned}$$

Observe that in finding the quotient of Z_1 and Z_2, we multiply Z_1 and Z_2 by the *conjugate* of Z_2 (i.e. the denominator). We describe this procedure as rationalizing the denominator of the expression.

Example 9.1.2

Find the product and quotient of the complex numbers $Z_1 = 5 + 4i, \ Z_2 = -2 + 3i$.

Solution

(i) The product of the complex numbers is given as;

$$\begin{aligned} Z_1 Z_2 &= (5+4i)(-2+3i) \\ &= -10+15i-8i-12 \\ &= -22+7i \end{aligned}$$

(ii) The quotient is given as;

$$\begin{aligned} \frac{Z_1}{Z_2} &= \frac{5+4i}{-2+3i} \\ &= \frac{5+4i}{-2+3i} \times \frac{-2-3i}{-2-3i} \\ &= \frac{-10-15i-8i+12}{4+6i-6i+9} \\ &= \frac{2-23i}{13} \\ &= \frac{1}{13}(2-23i) \end{aligned}$$

9.2 Conjugate of a Complex Number

If $Z = x + iy$, then the complex conjugate of Z is the complex number denoted by $\overline{Z}$ and is given as $\overline{Z} = x - iy$; (x and y being real). Two complex numbers which differ from each other only by the sign of the imaginary part are called the conjugate complex numbers. For example, the complex conjugate of $Z = 3 + 5i$ is given as $\overline{Z} = 3 - 5i$.

9.2.1 Properties of Complex Conjugates

Let $Z = x + iy$. Then,

(i) Rule I:

$$\begin{aligned} Z+\overline{Z} &= (x+iy)+(x-iy) \\ &= 2x \\ x &= \frac{Z+\overline{Z}}{2} \\ &= \frac{1}{2}(Z+\overline{Z}) \end{aligned}$$

It follows that,

$$ReZ = \frac{1}{2}(Z+\overline{Z})$$

(ii) Rule II:

$$\begin{aligned} Z-\overline{Z} &= (x+iy)-(x-iy) = 2iy \\ y &= \frac{1}{2i}(Z-\overline{Z}) \end{aligned}$$

So that $ImZ = \frac{1}{2i}(Z-\overline{Z})$

(iii) Rule III:

$$\overline{Z_1+Z_2} = \overline{Z_1}+\overline{Z_2}$$

(iv) Rule IV:

$$\overline{Z_1.Z_2} = \overline{Z_1}.\overline{Z_2}$$

9.3 Absolute Value of a Complex Number

We now generalize the concept of absolute value of complex number system.

Definition 14 *If $Z = x+iy$, we define the modulus, or absolute value, of Z as the non-negative real number $|Z|$ given by,*

$$|Z| = \sqrt{x^2+y^2}$$

For example, if $Z = 3-5i$, then,

$$|Z| = \sqrt{3^2+5^2} = \sqrt{34}$$

9.3.1 Properties of the Modulus of a Complex Number

If $Z_1, Z_2, \ldots, Z_n$ are complex numbers, the following properties hold:

(i) $|\overline{Z}| = |Z|$

(ii) $|Z_1 Z_2| = |Z_1||Z_2|$ or $\left|\frac{Z_1}{Z_2}\right| = \frac{|Z_1|}{|Z_2|}$, provided $Z_2 \neq 0$

(iii) $|Z_1 + Z_2| \leq |Z_1| + |Z_2|$

(iv) $|Z_1 - Z_2| \geq |Z_1| - |Z_2|$

We shall prove (iii) and leave (iv) as an exercise.

Proof of (iii). To show that given any complex number say Z_1 and Z_2,

$$|Z_1 + Z_2| \leq |Z_1| + |Z_2|$$

Now using (i),

$$\begin{aligned} |Z_1 + Z_2|^2 &= (Z_1 + Z_2)\overline{(Z_1 + Z_2)} \\ &= (Z_1 + Z_2)(\overline{Z_1} + \overline{Z_2}) \end{aligned}$$

So that,

$$\begin{aligned} |Z_1 + Z_2|^2 &= Z_1\overline{Z}_1 + Z_1\overline{Z}_2 + Z_2\overline{Z}_1 + Z_2\overline{Z}_2 \\ &\leq |Z_1|^2 + 2|Z_1 Z_2| + |Z_2|^2 \\ &= (|Z_1| + |Z_2|)^2 \end{aligned}$$

Taking square root of both sides:

$$|Z_1 + Z_2| \leq |Z_1| + |Z_2|$$

which completes the proof. □

9.4 Exponential Form of Complex Numbers

The exponential e^x (x being real) was mentioned in Section (1.4) of Chapter 1. Conversely, in this chapter we wish to define e^z,

Z being a complex number. We shall do this in a such a way that the principal properties of the real exponential function will be preserved. Recall that the main properties of e^x for x real are given by the law of exponents (Equation 4.1.1), $e^x e^y = e^{x+y}$, and the fact that $e^\circ = 1$. We shall give a definition of e^Z for complex Z which will preserve these properties and which will reduce to the ordinary exponential when Z is real.

If we write $Z = x+iy$, (x, y being real), then in order for the law of exponents to hold, we want $e^{x+iy} = e^x e^{iy}$. It therefore remains to define what we mean by e^{iy}.

Definition 15 *If $Z = x+iy$, we define $e^Z = e^{x+iy}$ to be the complex number:*

$$e^Z = e^x(\cos y + i \sin y) \tag{9.4.1}$$

If $x = 0$ and $y = \theta$, then,

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{9.4.2}$$

This is Euler formula.

This definition clearly agrees with the usual exponential function when Z is real (that is, $y = 0$). We shall now prove that the law of exponents still holds.

Theorem 9.4.1. *Suppose $Z_1 = x_1 + iy_1$, and $Z_2 = x_2 + iy_2$ are two complex numbers, then we have,*

$$e^{Z_1} e^{Z_2} = e^{Z_1+Z_2}$$

Proof.

$$e^{Z_1} = e^{x_1}(\cos y_1 + i\sin y_1);\ e^{Z_2} = e^{x_2}(\cos y_2 + i\sin y_2)$$

Thus,

$$\begin{aligned} e^{z_1}e^{z_2} &= e^{x_1}e^{x_2}[\cos y_1 \cos y_2 + i \sin y_2 \cos y_1 + i \sin y_1 \cos y_2 \\ &\quad +i^2 \sin y_1 \sin y_2] \\ &= e^{x_1}e^{x_2}[\cos y_1 \cos y_2 - \sin y_1 \sin y_2 \\ &\quad +i(\cos y_1 \sin y_2 + \sin y_1 \cos y_2)] \end{aligned}$$

Now, $e^{x_1}e^{x_2} = e^{x_1+x_2}$, and since x_1 and x_2 are both real, we have,

$$\cos y_1 \cos y_2 - \sin y_1 \sin y_2 = \cos(y_1 + y_2)$$

and,

$$\cos y_1 \cos y_2 + \sin y_1 \cos y_2 = \sin(y_1 + y_2)$$

Hence,

$$\begin{aligned} e^{Z_1}e^{Z_2} &= e^{Z_1+Z_2}[\cos(y_1 + y_2 + i \sin(y_1 + y_2)] \\ &= e^{Z_1+Z_2} \end{aligned} \tag{9.4.3}$$

□

9.4.1 Properties of Complex Exponentials

We shall now derive other properties of the complex exponential. Let $Z_1, Z_2, Z_3, \ldots, Z_n$ denote complex numbers, then the following properties hold:

(i) e^Z can not be zero.

Proof.

$$\begin{aligned} e^Z e^{-Z} &= e^{Z-Z} \\ &= e^{\circ} \\ &= 1 \end{aligned}$$

Hence e^Z cannot be zero. □

(ii) If y is real, then $|e^{iy}| = 1$

Proof.

$$\begin{aligned} |e^{iy}|^2 &= |\cos y + i \sin y|^2 \\ &= \left(\sqrt{(\cos y)^2 + (\sin y)^2}\right)^2 \\ &= \cos^2 y + \sin^2 y \\ &= 1 \end{aligned}$$

and $|e^{iy}| > 0$. □

(iii) $e^Z = 1$ if, and only if Z is an integral multiple of $2\pi i$.

Proof. Suppose $Z = 2\pi i n$, where n is an integer, then,

$$e^Z = \cos(2\pi n) + i\sin(2\pi n) = 1$$

Conversely, suppose that $e^z = 1$. This means that $e^x \cos y = 1$ and $e^x \sin y = 0$.
Since $e^x \neq 0$, we must have $\sin y = 0$; $y = k\pi$ where k is an integer. But $\cos(k\pi) = (-1)^k$.
Hence $e^x = (-1)^k$, since $e^x \cos(k\pi) = 1$. Since $e^x > 0$, k must be even. Therefore, $e^x = 1$ and hence $x = 0$. This proves the assertion. □

(iv) $e^{Z_1} = e^{Z_2}$ if, and only if, $Z_1 - Z_2 = 2\pi i n$, (where n is an integer).

Proof. $e^{Z_1} = e^{Z_2}$ if, and only if,

$$e^{Z_1 - Z_2} = 1$$

□

9.5 Geometric Representation of Complex Numbers

Just as real number are represented geometrically (Section 1.2.2) by points on a *Cartesian plane*, so likewise, complex numbers are represented by points in an *Argand plane*. The complex number $Z = (x, y)$ can be thought of as the 'point' with coordinates (x, iy).

From Figure 9.1, every complex number $Z = (x + iy)$ can be represented by a point $Z = (x, y)$ of the coordinate plane such that the *abscissa* is equal to the real part of the complex number, and its *ordinate* to the imaginary part. In this case, the coordinate plane is called the *complex plane*. The axis of abscissa is then called the *real axis* since it incorporates the points corresponding to the complex

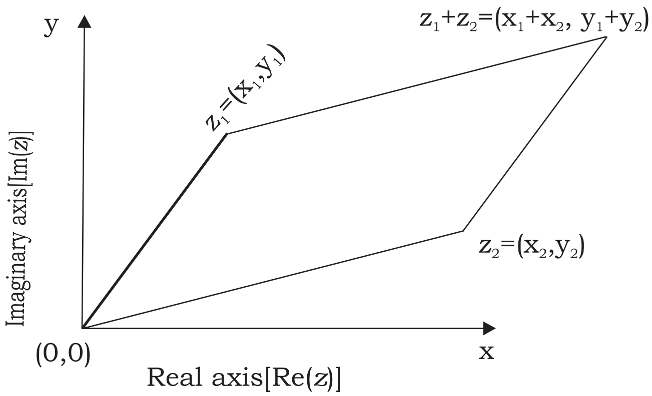

Figure 9.1: Addition of Complex numbers

number $x + oi$, that is, to real numbers. While the axis of ordinates is called the *imaginary axis*: lying on it are the points corresponding to *pure imaginary complex number* $0 + iy$.

9.5.1 The Argument of a Complex Number

Complex numbers Z, having one and the same modulus $|Z| = r$ correspond to the points of the complex plane lying on the circle of radius $\underline{r}$ with center at the point $Z = 0$, that is at the point O (Figure 9.2).

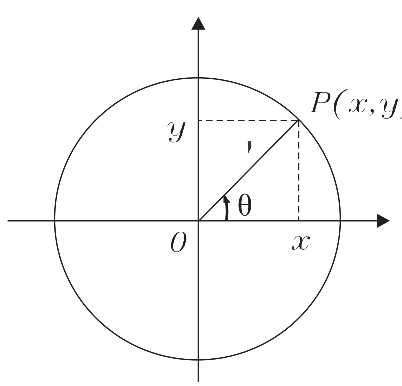

Figure 9.2: Argand diagram

If $|Z| \neq 0$, then infinitely many number exist with the given modulus. Only one complex number, namely $Z = 0$ has a modulus equal to zero. Geometrically, it is obvious that in order to single a concrete number out of the set of complex numbers with a given modulus $|Z| = r \neq 0$, it suffices to specify the direction of the vector $\vec{OP}$ i.e. (to specify the angle as in Figure 9.2)

Obviously, the point $Z = (x, y) = x + iy$ can be represented by *polar coordinates* r and θ, (Figure 9.2) and we can write,

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta \tag{9.5.1}$$

So that,

$$Z = r\cos\theta + ir\sin\theta = re^{i\theta} \tag{9.5.2}$$

and

$$\bar{Z} = r\cos\theta - i\sin\theta = re^{-i\theta} \tag{9.5.3}$$

The two numbers $\underline{r}$ and $\underline{\theta}$ uniquely determine Z. Conversely, the positive number r is uniquely determined by Z; since $r = |Z|$. However, Z determines the angle θ only up to multitudes of 2π. There are infinitely many values of θ which satisfy the equations

$$x = |Z|\cos\theta, \quad y = |Z|\sin\theta \tag{9.5.4}$$

but, of course any two of them differed by some multiples of 2π. Each of such θ is called an *argument of* Z but one of these values is singled out and is called the *principal argument of* Z. The angle is regarded as positive if it is measured anticlockwise, and as negative if it is measured clockwise.

Definition 16 *Let $Z = x + iy$ be a non-zero complex number. The angle θ between the positive direction of the real axis and the radius vector $\vec{OP}$ with the origin at the point $O(0, 0)$ and the terminus at the point $P(x, y)$ which satisfies the conditions;*

$$x = |Z|\cos\theta, \quad y = |Z|\sin\theta; \quad -\pi < \theta < +\pi$$

is called the principal argument (or simply, argument) of Z, denoted by,

$$\theta = arg(Z) \quad or \quad \arg(x + iy) \tag{9.5.5}$$

The above discussion immediately yields the following remark:

Remark 21 *Every complex number $Z \neq 0$ can be represented in the form*

$$Z = re^{i\theta} \quad \text{and} \quad \overline{Z} = e^{-i\theta} \tag{9.5.6}$$

where $r = |Z|$ and $\theta = \arg(Z) + 2\pi n$; n being any integer.

Observe that this method of representing complex numbers is particularly useful in connection with multiplication and division since we have;

$$Z_1 Z_2 = (r_1 e^{i\theta_1})(r_2 e^{i\theta_2}) = r_1 r_2 e^{(i\theta_1 + \theta_2)} \tag{9.5.7}$$

and

$$\frac{Z_1}{Z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)} \tag{9.5.8}$$

Example 9.5.1

Given the modulus of a complex number $r = 4$ and its argument $\theta = \frac{\pi}{6}$. Find the real and imaginary part of the complex number.

Solution

By (9.5.4), we have the real part,

$$\begin{aligned} x &= |Z| \cos\theta \\ &= 4\cos 30^\circ \\ &= 4\frac{\sqrt{3}}{2} \\ &= 2\sqrt{3} \end{aligned}$$

and imaginary part,

$$\begin{aligned} y &= |Z| \sin\theta \\ &= 4\sin 30^\circ \\ &= 4\left(\frac{1}{2}\right) = 2 \end{aligned}$$

Example 9.5.2

Find the argument of each of the following complex numbers and hence write down in polar form: (i) $Z = i$; (ii) $Z = 1 - i$;

(iii) $Z = -7$; (iv) $Z = 2 + 3i$; (v) $-\sqrt{3} + i$;
(vi) $-1 - i = Z$

Solution

(i) We first compute the modulus of the number Z, such that,

$$|Z| = r = \sqrt{0 + 1^2} = 1$$

Now, we find the argument of the number Z whose real part is equal to 0. Making use of (9.5.4) and the found value of the modulus, we obtain:

$$\begin{aligned} 0 = r\cos\theta &= 1\cos\theta \\ \text{i.e.,}\quad \cos\theta &= 0 \\ \therefore \quad \theta &= 90^\circ = \tfrac{\pi}{2} + 2\pi n;\ n \in Z \end{aligned}$$

Therefore by (9.5.2), we write the polar form of Z as:

$$\begin{aligned} Z &= r\cos\theta + ir\sin\theta \\ &= 1\cos\frac{\pi}{2} + i1\sin\frac{\pi}{2} \\ &= \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} \\ &= e^{\frac{\pi}{2}i} \end{aligned}$$

(ii) $Z = 1 - i$, so that the modulus:

$$|Z| = r = \sqrt{1^2 + 1^2} = \sqrt{2}$$

Thus, with the real part equal to unity, we have;

$$\begin{aligned} 1 &= r\cos\theta \\ &= \sqrt{2}\cos\theta \\ \text{i.e., } \cos\theta &= \frac{1}{\sqrt{2}} \\ &= \frac{\sqrt{2}}{2} \\ \therefore \qquad \arg(Z) &= \theta = 45^\circ = \frac{\pi}{4} - 2\pi n; \quad n \in Z \end{aligned}$$

Then the polar form of Z is;

$$\begin{aligned} Z &= r\cos(-\frac{\pi}{4}) + ir\sin(-\frac{\pi}{4}) \\ &= re^{-\pi/4i} \\ &= \sqrt{2}(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}) \\ &= \sqrt{2}e^{\frac{-\pi i}{4}} \end{aligned}$$

(iii) $Z = -7$. i.e.,

$$|Z| = \sqrt{7^2 + 0^2} = 7 = r$$

Since the real part of $Z = -7$ then,

$$\begin{aligned} -7 &= r\cos\theta \\ &= 7\cos\theta \\ \text{i.e.,} \quad \cos\theta &= \frac{-7}{7} \\ &= -1 \\ \therefore \quad \arg(Z) &= \theta = 180^\circ = \pi + 2\pi n;\ n \in Z \end{aligned}$$

Thus, the polar form of Z is given as:

$$\begin{aligned} Z &= r\cos\pi + ir\sin\pi &= re^{i\pi} \\ &= 7(\cos\pi + i\sin\pi) &= 7e^{i\pi} \end{aligned}$$

(iv) $Z = 2 + 3i$, i.e.,

$$|Z| = r = \sqrt{2^2 + 3^2} = \sqrt{13}$$

Thus, with the real part $x = 2$, we have;

$$\begin{aligned} 2 &= r\cos\theta \\ &= \sqrt{13}\cos\theta \\ \text{i.e.,} \quad \cos\theta &= \frac{2}{\sqrt{13}} \\ &= \frac{2\sqrt{13}}{13} \\ \therefore \quad \arg(Z) &= \theta = 56^\circ = \frac{14}{45}\pi + 2\pi n;\ n \in Z \end{aligned}$$

Therefore the polar form of Z is;

$$\begin{aligned} Z &= r\cos\frac{14}{45}r + ir\sin\frac{14}{45}\pi \\ &= re^{\frac{14}{45}i} \\ &= \sqrt{13}(\cos\frac{14}{45}\pi + i\sin\frac{14}{45}\pi) \\ &= \sqrt{13}e^{\frac{14}{45}i} \end{aligned}$$

(v) $Z = -\sqrt{3} + i$, i.e.,

$$|Z| = r = \sqrt{(\sqrt{3})^2 + 1^2} = 2$$

i.e.,

$$\begin{aligned} -\sqrt{3} &= r\cos\theta \\ &= 2\cos\theta \\ \text{i.e.,} \quad \cos\theta &= -\frac{\sqrt{3}}{2} \\ \therefore \quad \arg(Z) &= \theta = 150^\circ = \frac{5}{6}\pi + 2\pi n;\ n \in Z \end{aligned}$$

and the polar form is;

$$\begin{aligned} Z &= r\cos\frac{5}{6}\pi + ir\sin\frac{5}{6}\pi \\ &= re^{\frac{5}{6}\pi i} \\ &= 2(\cos\frac{5}{6}\pi + i\sin\frac{5}{6}\pi) \\ &= 2e^{\frac{5\pi i}{6}} \end{aligned}$$

(vi) $Z = -1 - i$, i.e.,

$$|Z| = r = \sqrt{1^2 + 1^2} = \sqrt{2}$$

So that,

$$\begin{aligned} -1 &= r\cos\theta \\ &= \sqrt{2}\cos\theta \\ \text{i.e.,} \qquad \cos\theta &= \frac{-1}{\sqrt{2}} \\ &= -\frac{\sqrt{2}}{2} \\ \therefore \qquad \arg(Z) &= \theta = 135^\circ = \frac{3}{4}\pi - 2\pi k;\ k \in Z \end{aligned}$$

and the polar form of Z is;

$$\begin{aligned} Z &= r\cos(-\frac{3}{4}\pi) + ir\sin(-\frac{3}{4}\pi) \\ &= re^{-\frac{3}{4}\pi i} \\ &= \sqrt{2}(\cos\frac{3}{4}\pi - i\sin\frac{3}{4}\pi) \\ &= \sqrt{2}e^{-\frac{3}{4}\pi i} \end{aligned}$$

Remark 22 *If* $Z = x + iy$, $\arg(Z)$ *can also be calculated thus;*

(i) $\arg(Z) = \tan^{-1}(\frac{y}{x})$, *if* $x > o$

(ii) $\arg(Z) = \tan^{-1}(\frac{y}{x}) + \pi$, *if* $x < 0$, $y \geq 0$,

(iii) $\arg(Z) = \tan^{-1}(\frac{y}{x}) - \pi$, *if* $x < 0$, $y < 0$

(iv) $\arg(Z) = \frac{\pi}{2}$, *if* $x = 0$, $y > 0$
and,

$$\arg(Z) = \frac{-\pi}{2} \quad \text{if } x = 0, \quad y < 0$$

Remark 23 *Please note that in the remaining part of this section, we shall restrict our use to lower case of* Z*, for instance, we shall be writing* $\arg(z)$ *to mean the same thing as* $\arg(Z)$*.*

Theorem 9.5.1. *If* $z_1 z_2 \neq 0$, *then,*

$$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2) + \arg(z_2) + 2\pi k(z_1, z_2)$$

where,

$$k(z_1, z_2) = \begin{cases} 0, \ \textit{if} \ -\pi < \arg(z_1) + \arg(z_2) \leq +\pi \\ +1, \ \textit{if} \ -2\pi < \arg(z_1) + \arg(z_2) \leq -\pi \\ -1, \ \textit{if} \ \pi < \arg(z_1) + \arg(z_2) \leq 2\pi \end{cases}$$

Proof. Write $z_1 = |z_1|e^{i\theta_1}$, and $z_2 = |z_2|e^{i\theta_2}$
where $\theta_1 = \arg(z_1)$, and $\theta_2 = \arg(z_2)$
Then,

$$z_1 z_2 = |z_1 z_2| e^{i(\theta_1+\theta_2)} \tag{9.5.9}$$

Since $-\pi < \theta_1 \leq +\pi$ and $-\pi < \theta_2 \leq +\pi$, we have;

$$-2\pi < \theta_1 + \theta_2 \leq 2\pi$$

Hence there is an integer k such that,

$$-\pi < \theta_1 + \theta_2 + 2\pi k \leq \pi$$

This k is the same as the integer $k(z_1, z_2)$ given in the theorem, and for this k we have,

$$\arg(z_1 z_2) = \theta_1 + \theta_2 + 2\pi k \tag{9.5.10}$$

which completes the proof of the theorem. □

Theorem 9.5.2: De Moivre's Theorem. *Suppose that z is any complex number and n is any integer, then,*

$$z^n = [r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta) \tag{9.5.11}$$

Proof.

(i) For any positive integer number n, this formula can be proved by induction:
For $n = 1$, the formula holds true. We assume that (9.5.11) is valid for $n = k$, such that the following equality holds true:

$$z^k = [r(\cos\theta + i\sin\theta)]^k = r^k(\cos k\theta + i\sin k\theta) \tag{9.5.12}$$

We shall prove that the validity of (9.5.12) implies that formula (9.5.12) holds true for $n = k + 1$ as well. Using formula (9.5.12), the operations involving complex numbers, the formulae for sine and the cosine of the sum of two angles or complex exponential properties (Section 9.4.2) we have;

$$\begin{aligned} z^{k+1} = z^k z &= [r^k(\cos k\theta + i\sin k\theta)][r(\cos\theta + i\sin\theta)] \\ &= r^{k+1}[(\cos k\theta\cos\theta - \sin k\theta\sin\theta) \\ &\quad +i(\sin k\theta\cos\theta + \cos k\theta\sin\theta)] \\ &= r^{k+1}[\cos(k+1)\theta + i\sin(k+1)\theta] \end{aligned}$$

Thus, we have proved formula (9.5.11) for $n = k + 1$. Consequently, using induction, we find that formula (9.5.11) is valid for any positive integer n.

(ii) If $n = 0$ and $z \neq 0$, then;

$$\begin{aligned} z^\circ &= 1 \text{ by definition} \\ \therefore \quad z^\circ &= 1(\cos 0\theta + i\sin 0\theta) \end{aligned}$$

Thus, formula (9.5.12) is valid for $n = 0$.

(iii) Let $n = -1$. Using the definition of a power with negative integral exponent and equation (9.5.6), we find that,

$$\begin{aligned} z^{-1} &= \frac{\cos 0 + i\sin 0}{r(\cos\theta + i\sin\theta)} \\ &= \frac{1}{r}[\cos(-\theta) + i\sin(-\theta)] \end{aligned}$$

Thus, formula (9.5.11) is valid for $n = -1$.

(iv) Suppose n is a negative integer, such that $n = -m$, where $m = |n|$ is a positive integer. Now using first the definition of a power with an integral exponent and then the validity of formula (9.5.12) first for $n = -1$, and then for positive integer

m, we have;

$$\begin{aligned} z^n = \left(\frac{1}{z}\right)^m &= \left\{\frac{1}{r[\cos(+\theta) + i\sin(+\theta)]}\right\}^n \\ &= \left\{\frac{1}{r}[\cos(-\theta) + i\sin(-\theta)]\right\}^m \\ &= \left(\frac{1}{r}\right)^m [\cos(-m\theta) + i\sin(-m\theta)] \\ &= r^n(\cos n\theta + i\sin n\theta)] \end{aligned} \tag{9.5.13}$$

Thus formula (9.5.12) holds true for any integer n, which completes the proof of the theorem.

□

Example 9.5.3

Find the 5^{th} power of the complex number $z = -\sqrt{3} + i$

Solution

We first express $-\sqrt{3} + i$ in polar form: Thus,

$$|z| = r = \sqrt{(\sqrt{3})^2 + 1^2} = 2$$

hence,

$$\begin{aligned} -\sqrt{3} &= r\cos\theta \\ &= 2\cos\theta \\ \text{i.e.,} \qquad \cos\theta &= -\frac{\sqrt{3}}{2} \\ \theta &= 150^\circ \\ \therefore \qquad \arg(z) &= \frac{5}{6}\pi + 2\pi k;\ k \in z \end{aligned}$$

Therefore, in polar form we have;

$$z = 2(\cos\frac{5}{6}\pi + i\sin\frac{5}{6}\pi)$$

So that

$$\begin{aligned} z^5 &= 2^5(\cos\frac{5}{6}\pi + i\sin\frac{5}{6}\pi)^5 \\ &= 2^5(\cos\frac{25}{6}\pi + i\sin\frac{25}{6}\pi) \\ &= 2^5 e^{\frac{25}{6}\pi i} \\ &= 32(\frac{\sqrt{3}}{2} + i\frac{1}{2}) = 16(\sqrt{3}+i) \end{aligned}$$

Example 9.5.4

Evaluate $(1-i\sqrt{3})^{\frac{-2}{5}}$ and leave your solution in polar form.

Solution

$$|z| = r = \sqrt{1^2 + (\sqrt{3})^2} = 2$$

Thus,

$$\begin{aligned} 1 &= r\cos\theta \\ &= 2\cos\theta \\ \text{i.e.,} \qquad \cos\theta &= \frac{1}{2} \\ \theta &= 60^\circ \\ \therefore \qquad \arg(z) &= \frac{1}{3}\pi - 2\pi k;\ k \in z \end{aligned}$$

Therefore in polar form;

$$z = 2(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3})$$

So that,

$$\begin{aligned} (1-i\sqrt{3})^{-2/5} &= 2^{-2/5}(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3})^{-2/5} \\ &= 2^{-2/5}(\cos(\frac{\pi}{3}.\frac{2}{5}) - i\sin(\frac{\pi}{3}.\frac{-2}{5})] \\ &= 2^{-2/5}[\cos(\frac{-2\pi}{15}) - i\sin(\frac{-2\pi}{15})] \\ &= 2^{-2/5}(\cos\frac{2\pi}{15} + i\sin\frac{2\pi}{15}) \end{aligned}$$

9.5.2 Roots of Complex Numbers

Theorem 9.5.3. *Suppose z is a complex number; $z \neq 0$, and there exists a positive integer n so that, there are n different complex numbers $w_1, w_2, \ldots, w_n$ such that $w_i^n = z$; where $i = 1, 2, \ldots, n$.*

These numbers w_i are called the n^{th} root of the complex number z of degree n. Observe that these roots are distinct from arithmetic roots, as there are no special symbols to designate them.

Proof. For $n = 1$, the theorem is obvious. But, suppose that $n \geq 2$ and

$$z = r(\cos\theta + i\sin\theta); \quad z \neq 0$$

We shall seek a complex number

$$w = p(\cos\theta + i\sin\theta)$$

such that $w^n = z$.

Now, let us show that such a number w does exist. We shall even prove that there are infinitely many numbers of that kind, but there are only n distinct numbers among them.

From De Moivre's Theorem 9.5.2,

$$z = w^n = p^n(\cos n\phi + i\sin n\phi)$$

By the definition of the absolute value of a complex number; $|z| = p^n$, i.e.

$$\begin{aligned} r &= p^n \\ \text{whence} \quad p &= \sqrt[n]{r} \end{aligned}$$

(according to the definition of the absolute value of the complex number $z \neq 0$, the numbers p and r are positive and therefore the use of the symbol of an arithmetic root is correct here).

Using the definition of the equality of two complex numbers, we find that the following equalities simultaneously holds true:

$$\left\{ \begin{array}{rcl} \cos n\phi &=& \cos\theta \\ \sin n\phi &=& \sin\theta \end{array} \right\} \tag{9.5.14}$$

These equalities are simultaneously satisfied if and only if,

$$n\phi = \theta + 2\pi k \qquad (9.5.15)$$

where k is any integer,

$$\text{i.e. for} \quad \phi = \frac{\theta + 2\pi k}{n}; \quad k \in z \qquad (9.5.16)$$

This means that there are numbers w such that each of them satisfies the equality,

$$w^n = r(\cos\theta + i\sin\theta)$$

and can be written in the form:

$$w = r^{1/n}\left[\cos\left(\frac{\theta + 2\pi k}{n}\right) + i\sin\left(\frac{\theta + 2\pi k}{n}\right)\right] \qquad (9.5.17)$$

where k is an integer. Designating as ω_m the root calculated from formula (9.5.17) for $k = m$, we get;

$$\begin{aligned}
w_0 &= r^{1/n}\left[\cos\left(\frac{\theta}{n}\right) + i\sin\left(\frac{\theta}{n}\right)\right]\\
w_1 &= r^{1/n}\left[\cos\left(\frac{\theta + 2\pi}{n}\right) + i\sin\left(\frac{\theta + 2\pi}{n}\right)\right]\\
w_2 &= r^{1/n}\left[\cos\left(\frac{\theta + 2\pi 2}{n}\right) + i\sin\left(\frac{\theta + 2\pi 2}{n}\right)\right]\\
&\vdots\\
&\vdots \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\vdots\\
w_n &= r^{1/n}\left[\cos\left(\frac{\theta + 2\pi n}{n}\right) + i\sin\left(\frac{\theta + 2\pi n}{n}\right)\right] = w_0\\
w_{n+1} &= r^{1/n}\left[\cos\left(\frac{\theta + 2\pi(n+1)}{n}\right) + i\sin\left(\frac{\theta + 2\pi(n+1)}{n}\right)\right]\\
&= w_1\\
&\vdots\\
&\vdots \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}$$

Thus,

$$\begin{aligned} w_{2n} &= w_0 \\ w_{-1} &= w_{n-1} \\ w_{-2} &= w_{n-2} \\ &\vdots \\ \dots & \quad \dots \\ w_{-n} &= w_0 \end{aligned}$$

Hence it is easy to see that for any integral <u>m</u> the equalities,

$$w_0 = w_{mn}, w_1 = w_{mn+1}, w_2 = w_{mn+2}, \dots w_{n-1} = w_{mn+n-1}$$

hold true.

Consequently, there are exactly <u>n</u> distinct roots:

$$w_o, w_1, w_2, \dots, w_{n-1}$$

These roots can be obtained by the formula:

$$w_k = r^{1/n}\left[\cos\left(\frac{\theta + 2\pi k}{n}\right) + i\sin\left(\frac{\theta + 2\pi k}{n}\right)\right] \tag{9.5.18}$$

where $k = 0, 1, 2, \dots, (n-1)$.

Observe, that we can obtain all the <u>n</u> distinct roots from formula (9.5.12) if we substitute in it any n successive integer k. □

Example 9.5.5

Find the roots of the following equations:
(i) $z^2 = i$, (ii) $z^3 = 5 - 12i$, (iii) $z^3 + 1 = 0$,
(iv) $z^3 = 1 - i\sqrt{3}$.

Solution

(i) since $r = 1$ and $\theta = \frac{\pi}{2}$, it follows that the polar form of z is,

$$z^2 = i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$$

Thus,

$$
\begin{aligned}
z &= \left[\cos\left(\frac{\pi}{2}+2\pi k\right)+i\sin\left(\frac{\pi}{2}+2\pi k\right)\right]^{\frac{1}{2}}\\
\therefore \quad z &= \cos\left(\frac{\frac{\pi}{2}+2\pi k}{2}\right)+i\sin\left(\frac{\frac{\pi}{2}+2\pi k}{2}\right);\ k=0,1
\end{aligned}
$$

Thus, for $k=0$;

$$
\begin{aligned}
z_0 &= \cos\left(\frac{\frac{\pi}{2}+0}{2}\right)+i\sin\left(\frac{\frac{\pi}{2}+0}{2}\right)\\
&= \cos\pi+i\sin\pi\\
&= -1
\end{aligned}
$$

For $k=1$;

$$
\begin{aligned}
z_1 &= \cos\left(\frac{\frac{\pi}{2}+2\pi}{2}\right)+i\sin\left(\frac{\frac{\pi}{2}+2\pi}{2}\right)\\
&= \cos\frac{5}{4}\pi+i\sin\frac{5}{4}\pi\\
&= \frac{-\sqrt{2}}{2}-\frac{i\sqrt{2}}{2}=-\frac{\sqrt{2}}{2}(1+i)
\end{aligned}
$$

Hence, the two roots are $-1, -\frac{\sqrt{2}}{2}(1+i)$.

(ii) $z^3=5-12i$
$r=\sqrt{5^2+12^2}=13$
Thus,

$$\phi=\arg(5-12i)+2\pi k; \quad k\in z$$

Therefore,

$$
\begin{aligned}
z &= 13^{1/3}[\cos(\phi-2\pi k)+i\sin(\phi-2\pi k)]^{1/3}\\
&= 13^{1/3}\left[\cos\left(\frac{\phi-2\pi k}{3}\right)+i\sin\left(\frac{\phi-2\pi k}{3}\right)\right]\\
&= 13^{1/3}\left[\cos\left(\frac{\phi+2\pi k}{3}\right)-i\sin\left(\frac{\phi+2\pi k}{3}\right)\right]^{1/3}
\end{aligned}
$$

where $k=0,1,2$.

(iii) $z^3 = -1$, i.e. $r = 1$, and $\theta = \pi$.
Hence the polar form of z is;

$$\begin{aligned} z^3 &= -1 \\ &= \cos\pi + i\sin\pi \\ z &= [\cos(\pi + 2\pi k) + i\sin(\pi + 2\pi k)]^{1/3} \\ &= \cos\left(\frac{\pi + 2\pi k}{3}\right) + i\sin\left(\frac{\pi + 2\pi k}{3}\right) \end{aligned}$$

where $k = 0, 1, 2$.

Thus, for $k = 0$,

$$\begin{aligned} z_0 &= \cos\frac{\pi}{3} + i\sin\frac{\pi}{3} \\ &= \frac{1}{2} + i\frac{\sqrt{3}}{2} = \frac{1}{2}(1 + i\sqrt{3}) \end{aligned}$$

For $k = 1$,

$$\begin{aligned} z_1 &= \cos\left(\frac{\pi + 2\pi}{3}\right) + i\sin\left(\frac{\pi + 2\pi}{3}\right) \\ &= \cos\pi + i\sin\pi \\ &= -1 \end{aligned}$$

For $k = 2$,

$$\begin{aligned} z_2 &= \cos\left(\frac{\pi + 4\pi}{3}\right) + i\sin\left(\frac{\pi + 4\pi}{3}\right) \\ &= \cos\frac{5}{3}\pi + i\sin\frac{5}{3}\pi \\ &= \frac{\sqrt{3}}{2} - \frac{1}{2}i = \frac{1}{2}(\sqrt{3} - i) \end{aligned}$$

Therefore, the three roots are:

$$\frac{1}{2}(1 + i\sqrt{3}),\ -1,\ \frac{1}{2}(\sqrt{3} - i)$$

(iv) $z^3 = 1 - i\sqrt{3}$, i.e., $r = 2$, and $\theta = \frac{\pi}{3}$.
Thus the polar form is;

$$\begin{aligned} z^3 &= 1 - i\sqrt{3} \\ &= 2(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}) \\ \therefore \qquad z &= 2^{1/3}\left[\cos\left(\frac{\frac{\pi}{3}+2\pi k}{3}\right) - i\sin\left(\frac{\frac{\pi}{3}+2\pi k}{3}\right)\right] \end{aligned}$$

9.5.3 The nth Roots of Unity

The solutions of the equation;

$$z^n = 1 \quad (\text{ i.e. } \theta = 0 \text{ and } r = 1)$$

where $\underline{n}$ is a positive integer are called the nth roots of unity and are given by,

$$w_k = \cos\left(\frac{2\pi k}{n}\right) + i\sin\left(\frac{2\pi k}{n}\right) = e^{\frac{2\pi k i}{n}} \tag{9.5.19}$$

where $k = 0, 1, 2, \ldots, (n-1)$.
If,

$$w = \cos\left(\frac{2\pi}{n}\right) + i\sin\left(\frac{2\pi}{n}\right) = e^{\frac{2\pi i}{n}} \tag{9.5.20}$$

then the n roots are;

$$1, w, w^2, \ldots, w^{n-1}$$

Note that the sum of these roots is zero and they represent the n vertices of a regular polygon of n sides inscribed in a circle of radius 1 with the center at the origin.

Properties of nth - Degree Root of Unity

(i) If $n = 2m$ (an even number), then there are two real roots $w_o = 1$ and $w_m = -1$ among these roots. Whereas, if

$$n = 2m + 1 \quad (\text{an odd number}) \;;$$

then there is one real root $w_o = 1$.

(ii) $|w_k| = 1$

(iii) $w_k w_m = w_{k+m}$

(iv) $\frac{w_k}{w_m} = w_{k-m}$.

(v) $w_k^m = w_{km}$,

Where m is an integer (the root w_n can be found from formula (9.5.19) where n must be taken instead of k)

We shall prove these properties as follows:

Proof.

(i) Property (i) is trivial, since it follows from the definition.

(ii) Also, property (ii) follows from the definition of the absolute values of a complex number (Section 9.3).

(iii) To prove property (iii), we make use of the theorem on the product of complex numbers in trigonometric form (Section 9.4.7):

$$\begin{aligned} w_k w_m &= \cos\left(\frac{2\pi k}{n} + \frac{2\pi m}{n}\right) + i\sin\left(\frac{2\pi k}{n} + \frac{2\pi m}{n}\right) \\ &= w_{k+m} \end{aligned} \tag{9.5.21}$$

(iv) Property (iv) can be proved by analogy.

(v) We shall prove property (v) by De Moivre's Theorem:

$$w_k^m = \cos\left[\left(\frac{2\pi k}{n}\right)m\right] + i\sin\left[\left(\frac{2\pi k}{n}\right)m\right] = w_{km} \tag{9.5.22}$$

Let us consider now the geometrical interpretation of the nth - degree of unity:

$$
\begin{aligned}
w_0 &= 1 \\
w_1 &= \cos\left(\frac{2\pi}{n}\right) + i\sin\left(\frac{2\pi}{n}\right) \\
w_2 &= \cos\left(\frac{4\pi}{n}\right) + i\sin\left(\frac{4\pi}{n}\right) \\
w_3 &= \cos\left(\frac{6\pi}{n}\right) + i\sin\left(\frac{6\pi}{n}\right) \\
\vdots & \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
w_{n-1} &= \cos\left[\frac{2\pi(n-1)}{n}\right] + i\sin\left[\frac{2\pi(n-1)}{n}\right]
\end{aligned}
$$

The points $w_0, w_1, w_2, \ldots, w_{n-1}$ are evidently the vertices of a regular n-sided polygon inscribed into a unit circle, one of whose vertices is the point $z_0(1,0)$.

□

Example 9.5.6

Find the third-degree roots of unity i.e. $z^3 = 1$.

Solution

We know already that $r = 1$ and $\theta = 0$; so that by the formula (9.5.19),

$$
\begin{aligned}
z_0 &= 1 \\
z_1 &= \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right) = -\frac{1}{2} + i\frac{\sqrt{3}}{2} \\
z_2 &= \cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right) = -\frac{1}{2} - i\frac{\sqrt{3}}{2}
\end{aligned}
$$

Observe that the points;

$$
z_o(1,0);\, z_1\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right);\, z_2\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)
$$

are the vertices of a regular triangle $z_0z_1z_2$ inscribed into a unit circle (Figure 9.3)

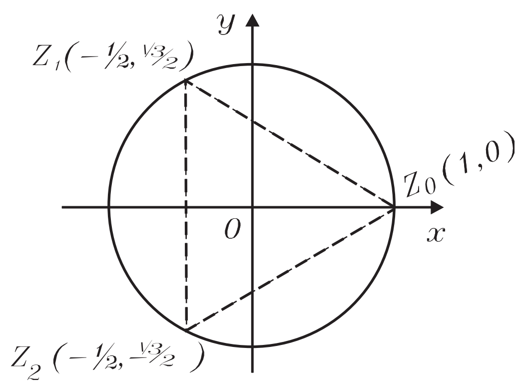

Figure 9.3: Vertices of a regular triangle inscribed in a unit circle

Example 9.5.7

Let $n = 4$
i.e.,

$$\begin{aligned}
z^4 &= 1 \\
z_0 &= 1 \\
z_1 &= \cos\left(\frac{2\pi}{4}\right) + i\sin\left(\frac{2\pi}{4}\right) = i \\
z_2 &= \cos\left(\frac{4\pi}{4}\right) + i\sin\left(\frac{4\pi}{4}\right) = -1 \\
z_3 &= \cos\left(\frac{6\pi}{4}\right) + i\sin\left(\frac{6\pi}{4}\right) = -i
\end{aligned}$$

Thus, the points;

$$z_0(1, 0); z_1(0, 1); z_2(-1, 0); z_3(0, -1)$$

are the vertices of the square:

$$z_0z_1z_2z_3$$

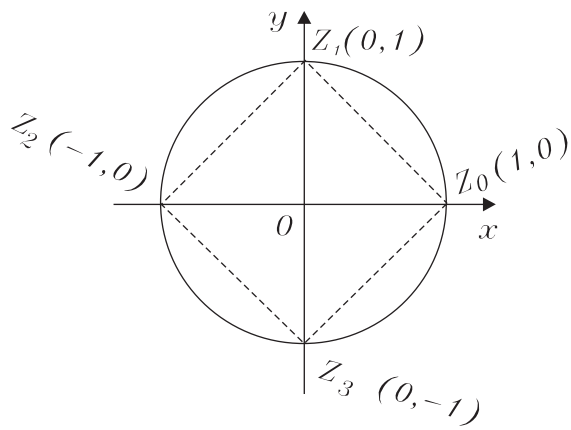

Figure 9.4: Vertices of a square inscribed in a unit circle

inscribed into a unit circle.

We shall give the general formula for the nth-degree root of the number, -1 i.e. $w^n = -1$. Here, we already know that $r = 1$, and $\theta = \pi$, it follows that;

$$w_k = \cos\left(\frac{\pi + 2\pi k}{n}\right) + i \sin\left(\frac{\pi + 2\pi k}{n}\right) \tag{9.5.23}$$

where $k = 0, 1, 2, \ldots, (n-1)$.

It could be observed that if $n = 2m$ (an even number), then there are no real roots among w_k and if $n = 2m + 1$ (an odd member), then there is one real number $w_m = -1$.

In general, for any positive number a and any even integer n, there are only two real numbers b_1 and b_2 such that,

$$b_1^n = b_2^n = a$$

Indeed, since $a = |a|(\cos 0 + i \sin 0)$ for any positive number, all the nth-degree roots of that number can be calculated by the formula;

$$b_k = (|a|)^{1/n}\left[\cos\left(\frac{2\pi k}{n}\right) + i \sin\left(\frac{2\pi k}{n}\right)\right] \tag{9.5.24}$$

If $n = 2m$, then there are only two real numbers

$$b_0 = (|a|)^{1/n} \text{ and } b_m = -(|a|)^{1/n}$$

among these numbers, and this was stated above.

By analogy, we shall prove the validity of the following statements.

(i) For any positive number a and any odd natural number n, there is only one real number.

$$b = \sqrt[n]{a}; \quad \text{such that } b^n = a$$

(ii) For any negative number a and any odd natural number n there is only one real number

$$b = -\sqrt[n]{|a|}; \quad \text{such that } b^n = a$$

(iii) For any negative number a and any even natural number n there is not a single real number b such that $b^n = a$.

9.5.4 Expression of powers of $\cos\theta$ and $\sin\theta$ in terms of multiple angles

Let $z = \cos\theta + i\sin\theta$ then we can write,

$$z^n = \cos(n\theta) + i\sin(n\theta)$$

Consequently,

$$\frac{1}{z} = \cos\theta - i\sin\theta$$

and

$$\frac{1}{z^n} = \cos(n\theta) - i\sin(n\theta). \tag{9.5.25}$$

So that,

$$z + \frac{1}{z} = 2\cos\theta, z - \frac{1}{z} = 2i\sin\theta \tag{9.5.26}$$

Similarly,

$$z^n + \frac{1}{z^n} = 2\cos(n\theta) \tag{9.5.27}$$

and

$$z^n - \frac{1}{z^n} = 2i\sin(n\theta) \tag{9.5.28}$$

The proof of these properties follows from the definition of De Moivre's Theorem 9.5.3.

Example 9.5.8

Establish the expression;

(i) $16\sin^5\theta = \cos 5\theta - 5\sin 3\theta + 10\sin\theta$

(ii) $32\cos^6\theta = \cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10$

Solution

(i) Now,

$$32i\sin^5\theta = (2i\sin\theta)^5 = \left(z - \frac{1}{z}\right)^5$$

Hence, expanding $(z-\frac{1}{z})^5$ by Binomial Theorem (7.4) we have;

$$\begin{aligned}
(z - \frac{1}{z})^5 &= (z^5 - \frac{1}{z^5}) - 5(z^3 - \frac{1}{z^3}) + 10(z - \frac{1}{z}) \\
&= 2i\sin 5\theta - 10i\sin 3\theta + 20i\sin\theta \\
&= 2i(\sin 5\theta - 5\sin 3\theta + 10\sin\theta) \\
\therefore \quad \frac{32i\sin^5\theta}{2i} &= \sin 5\theta - 5\sin 3\theta + 10\sin\theta \\
\text{or} \quad 16\sin^5\theta &= \sin 5\theta - 5\sin 3\theta + 10\sin\theta
\end{aligned}$$

(ii) Suppose,

$$64\cos^6\theta = (2\cos\theta)^6 = (z + \frac{1}{z})^6$$

Thus, expanding by Binomial (Section 7.4) we have;

$$\begin{aligned}
\left(z + \frac{1}{z}\right)^6 &= \left(z^6 + \frac{1}{z^6}\right) + 6\left(z^4 + \frac{1}{z^4}\right) \\
&\quad +15\left(z^2 + \frac{1}{z^2}\right) + 20 \\
&= 2\cos 6\theta + 12\cos 4\theta + 30\cos 2\theta + 20 \\
&= 2(\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10) \\
\therefore \quad \frac{64\cos^6\theta}{2} &= \cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10 \\
\text{or} \quad 32\cos^6\theta &= \cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10
\end{aligned}$$

Example 9.5.9

Use De Moivre's Theorem to derive the following trigonometric identities:

(i) $\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$

(ii) $\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$

Solution

By De-Moivre's Theorem 9.5.2, we have;

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$$

So that if $n = 3$ we have;

$$(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta$$

where $Re(z) = \cos 3\theta$ and $Im(z) = \sin 3\theta$.
Now, expanding $(\cos\theta + i\sin\theta)^3$ by the Binomial theorem, we obtain;

$$\begin{aligned}(\cos\theta + i\sin\theta)^3 &= \cos^3\theta + 3\cos^2\theta i\sin\theta + 3\cos\theta(i\sin\theta)^2 \\ &\quad +(i\sin\theta)^3 \\ &= \cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta \\ &= \cos^3\theta - 3\cos\theta\sin^2\theta + i(3\cos^2\theta\sin\theta - \sin^3\theta)\end{aligned}$$

Therefore,
Real part; $Re(z) = \cos^3\theta - 3\cos\theta\sin^2\theta$.
Imaginary part; $Im(z) = 3\cos^2\theta\sin\theta - \sin^3\theta$.
Thus,

$$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$$

and,

$$\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$$

Example 9.5.10

Use De Moivre's Theorem to find the values of; (i) $\cos 5\theta$,
(ii) $\sin 5\theta$

Solution

$$(\cos\theta + i\sin\theta)^5 = \cos 5\theta + i\sin 5\theta \quad \text{(DeMoivre's theorem)}$$

where $Re(z) = \cos 5\theta$ and $Im(z) = \sin 5\theta$.
Now, expanding by Binomial Theorem (section 7.4) we have;

$$\begin{aligned}(\cos\theta + i\sin\theta)^5 &= \cos^5\theta + 5\cos^4\theta i\sin\theta + 10\cos^3\theta(i\sin\theta)^2 \\ &\quad +10\cos^2\theta(i\sin\theta)^3 + 5\cos\theta(i\sin\theta)^4 \\ &\quad +(i\sin\theta)^5 \\ &= \cos^5\theta + 5i\cos^4\theta\sin\theta - 10\cos^3\theta\sin^2\theta \\ &\quad -10i\cos^2\theta\sin^3\theta + 5\cos\theta\sin^4\theta + i\sin^5\theta \\ &= \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta \\ &\quad +i(5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta)\end{aligned}$$

Thus,

$$\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$$

and

$$\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$$

(Note that $i^2 = -1, \quad i^3 = -i, \quad i^4 = +1, \quad i^5 = i$)

9.6 Complex Logarithms

We already know that e^z is never zero, for any complex number. It is natural to ask if there are other values that e^z cannot assume. We shall observe in the next theorem that zero is the only exceptional value.

Theorem 9.6.1. *If z is a complex number such that $z \neq 0$, then there exist complex numbers w such that;*

$$e^w = z \tag{9.6.1}$$

One such w is the complex number,

$$\log|z| + i\arg(z)$$

and any other such w must have the form;

$$\log|z| + i\arg(z) + 2\pi ni \quad (n \text{ is an integer })$$

Proof. Since,

$$e^{\log|z|+i\arg(z)} = e^{\log|z|}e^{i\arg(z)} = |z|e^{i\arg(z)} = z \tag{9.6.2}$$

we observe that,

$$w = \log|z| + i\arg(z) \tag{9.6.3}$$

is a solution of the equation $e^w = z$. But w_1 is any other solution, then $e^w = e^{w_1}$ and hence,

$$w - w_1 = 2\pi ni \tag{9.6.4}$$

□

Definition 17 *Suppose $z \neq 0$ is a given complex number. Let w be a complex number such that $e^w = z$, then w is called the logarithm of z. In particular the value of w as given by,*

$$w = \log|z| + i\arg(z)$$

is called the principal logarithm of z, and hence we can write,

$$w = \log z$$

Example 9.6.1

(i) If $z = i$, then $|z| = 1$ and $\arg(z) = \frac{\pi}{2}$,
Therefore,

$$\begin{aligned} w = \log(z) &= \log|z| + i\arg(z) \\ &= 0 + \frac{i\pi}{2} \\ &= \frac{\pi}{2}i \end{aligned}$$

(ii) $z = -i$, then $|z| = 1$ and $\arg(z) = -\frac{\pi}{2}$
Therefore,

$$\begin{aligned}\log|z| &= \log|-i| + i\arg(-i) \\ &= 0 + -i\frac{\pi}{2} \\ &= -\frac{\pi}{2}i\end{aligned}$$

(iii) $z = -1$, then $|z| = 1$ and $\arg(z) = \pi$.
Therefore,

$$\begin{aligned}\log(z) &= \log|-1| + i\arg(-1) \\ &= 0 + i\pi \\ &= \pi i\end{aligned}$$

(iv) If $x > 0$, then $\log(x) = \log x$, since $|x| = x$ and $\arg(x) = 0$.

(v) $z = 1 + i$, then $|z| = \sqrt{2}$, and $\arg(z) = \frac{\pi}{4}$,
Thus,

$$\begin{aligned}\log(z) &= \log|1+i| + i\arg(1+i) \\ &= \log\sqrt{2} + i\pi/4\end{aligned}$$

Theorem 9.6.2. *Let $z_1 z_2 \neq 0$, then,*

$$\log(z_1 z_2) = \log z_1 + \log z_2 + 2\pi i n(z_1 z_2)$$

where $n(z_1, z_2)$ is the integer defined in Theorem 9.6.1.

Proof.

$$\begin{aligned}\log(z_1 z_2) &= \log|z_1 z_2| + i\arg(z_1 z_2) \\ &= \log|z_1| + \log|z_2| + i[\arg(z_1) + \arg(z_2)] + 2\pi n(z_1, z_2)\end{aligned}$$

□

9.7 Complex Powers

Using logarithms, we shall now present a definition of complex powers of complex numbers as follows:

Definition 18 *Let* $z \neq 0$ *and suppose* w *is any complex number; we define,*

$$z^w = e^{w \log z} \tag{9.7.1}$$

Example 9.7.1

Show that; (i) $i = e^{\pi/2}$; (ii) $(-1)^i = e^{-\pi}$; (iii) $z^\rho = z^\rho z$ if ρ is an integer.

Solution

(i)

$$i^i = e^{i \log i} = e^{i(i\pi/2)} = e^{-\pi/2}$$

(ii)

$$(-1)^i = e^{i \log(-1)} = e^{i(i\pi)} = e^{-\pi}$$

(iii) If ρ is an integer, then;

$$\begin{aligned} z^{\rho+1} = e^{(\rho+1)\log z} &= e^{\rho \log z} e^{\log z} \\ &= e^{\log z^\rho} e^{\log z} = z^\rho z \end{aligned}$$

The next two theorems give us rules for calculating with complex powers.

Theorem 9.7.1.

$$z^{w_1} z^{w_2} = z^{w_1 + w_2}$$

Proof.

$$\begin{aligned} z^{w_1+w_2} &= e^{(w_1+w_2)\log z} \\ &= e^{w_1 \log z} e^{w_2 \log z} \\ &= z^{w_1} z^{w_2} \end{aligned}$$

Therefore,

$$z^{w_1+w_2} = z^{w_1} z^{w_2} \tag{9.7.2}$$

□

Theorem 9.7.2.

$$(z_1 z_2)^w = z_1^w z_2^w e^{2\pi i w n(z_1,z_2)}$$

where $n(z_1, z_2)$ is an integer.

Proof.

$$(z_1 z_2)^w = e^{w \log(z_1 z_2)} = e^{w[\log z_1 + \log z_2 + 2\pi i n(z_1,z_2)]} \tag{9.7.3}$$

□

9.7.1 Complex sines and cosines

Definition 19 *Let z be any given complex number, then we define,*

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}; \ \sin z = \frac{e^{iz} - e^{-iz}}{2i} \tag{9.8.1}$$

Theorem. *Let $z = x + iy$, then we have;*

$$\cos z = \cos x \cosh y - i \sin x \sinh y \tag{9.8.2}$$

$$\sin z = \sin x \cosh y + i \cos x sinhy \tag{9.8.3}$$

Proof.

$$\begin{aligned} 2\cos z &= e^{iz} + e^{-iz} \\ &= e^{-y}(\cos x + i \sin x) + e^{y}(\cos x - i \sin x) \\ &= \cos x(e^{y} + e^{-y}) - i \sin x(e^{y} - e^{-y}) \\ &= 2\cos x \cosh y - 2i \sin x \sinh y \end{aligned}$$

□

The proof for $\sin z$ is analogous.

Chapter 10

Matrix Algebra and Systems of Linear Equations

In Chapter 2, we dealt with how to solve linear equations involving one or two variables (simultaneous linear equations). However, many problems in mathematics, engineering and business lead not only to a single or two linear equations in one or two variables, but to a whole system of such equations. It is the drawback of the simultaneous methods in solving whole systems of linear equations that forms the basis of this chapter.

Therefore, this chapter deals with the methods of solving systems of m linear equations in n unknowns (i.e. $m \times n$ equations), where m and n become too large to solve simultaneously. The concepts of a matrix and its related operations prove convenient as they facilitate the manipulations and calculations involved in solving such systems of linear equations.

10.1 Definition of a Matrix

A set of numbers (real, rational, or complex) arranged in a rectangular array of m rows and n columns enclosed within curved or square brackets is called a *matrix* (of numbers). The rows and columns of equation 10.1 below are termed the lines of the matrix:

$$A = \begin{pmatrix} a_{11} & a_{12}...a_{1n} \\ a_{21} & a_{22}...a_{2n} \\ . & .... \\ a_{m1} & a_{m2}...a_{mn} \end{pmatrix} \text{ or } \begin{bmatrix} a_{11} & a_{12}...a_{1n} \\ a_{21} & a_{22}...a_{2n} \\ . & .... \\ a_{m1} & a_{m2}...a_{mn} \end{bmatrix} \qquad (10.1.1)$$

is a matrix of order $m \times n$.

The numbers $a_{ij}(i = 1, 2, ..., m;\ j = 1, 2, ..., n)$ that comprise the given matrix are called the *elements (or entries)* of the matrix. Here, the first subscript i denotes the row of the element and the second subscript j denotes the column of the element.

A matrix A, say, is often more compactly written as,

$$A = [a_{ij}]; \quad (i = 1, 2, ..., m;\ j = 1, 2, ..., n) \qquad (10.1.2)$$

$$\text{or} \quad A = [a_{ij}]m, n \qquad (10.1.3)$$

We say that the matrix A has *order* $m \times n$, or that A is an m-by-n matrix (or an $m \times n$ matrix, or is of type $m \times n$)

10.2 Types of Matrix

10.2.1 Row matrix

When $m = 1$, A is a row matrix; that is if the number of row is one. e.g.,

$$A = (a_1 a_2 a_3 ... a_n) \ \text{ or } \ [a_1 a_2 a_3 ... a_n] \qquad (10.2.1)$$

Equation 10.2.1 is called a *row matrix* or *row vector*.

10.2.2 Column Matrix

When $n = 1$, i.e. when the number of column is one; we have an $m \times 1$ matrix which is of the form:

$$A = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{pmatrix} \text{ or } \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix} \qquad (10.2.2)$$

which is called a *column matrix* or *column Vector*.

10.2.3 Square Matrix

If $m = n$, i.e. when the number of rows of any given matrix is equal to the number of columns , then such an $n \times n$ matrix is called a *Square matrix* of order n and is of the form:

$$A = \begin{bmatrix} a_{11} & a_{12}...a_{1n} \\ a_{21} & a_{22}...a_{2n} \\ . & .... \\ a_{n1} & a_{n2}...a_{nn} \end{bmatrix}; \; m = n \tag{10.2.3}$$

10.2.4 Rectangular Matrix

If $m \neq n$, i.e. when the number of rows of any given matrix is not equal to the number of columns, then such a matrix is called a *rectangular matrix* of order $m \times n$ and is of the form:

$$A = \begin{bmatrix} a_{11} & a_{12}...a_{1n} \\ a_{21} & a_{22}...a_{2n} \\ . & .... \\ a_{m1} & a_{m2}...a_{mn} \end{bmatrix}; \; m \neq n \tag{10.2.4}$$

10.2.5 Diagonal Matrix

A square matrix of order n such that all the non-diagonal elements are all zero is called a diagonal matrix of order n and is of the form:

$$A = \begin{bmatrix} a_{11} & 0 & 0 & ... & 0 \\ 0 & a_{22} & 0 & ... & 0 \\ ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & a_{nn} \end{bmatrix}; A_{ij} = 0 (i \neq j) \tag{10.2.5}$$

The elements $[a_{11}, a_{22}...a_{nn}]$ are called *diagonal elements.*

10.2.6 Identity Matrix

A diagonal matrix in which all the diagonal elements or entries are all equal to 1 is called an *identity matrix* or *unit matrix* and is denoted by the letter I; which is of the form:

$$I = \begin{bmatrix} 1 & 0 & 0 & ... & 0 \\ 0 & 1 & 0 & ... & 0 \\ ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & 1 \end{bmatrix} \qquad (10.2.6)$$

Thus we can write, $I = [\delta_{ij}]$; such that,

$$\delta_{ij} = \begin{cases} 0, & \text{if} \quad i \neq j \\ 1, & \text{if} \quad i = j \end{cases} \qquad (10.2.7)$$

10.2.7 Triangular Matrix

This type of matrix is subdivided into two; namely, upper triangular matrix and lower triangular matrix. As the name implies, the *upper triangular matrix* is a square matrix in which all the elements or entries below the diagonal are all zero, and is of the form:

$$A = \begin{bmatrix} a_{11} & a_{12} & ... & a_{1n} \\ 0 & a_{22} & ... & a_{2n} \\ ... & ... & ... & ... \\ 0 & 0 & ... & a_{nn} \end{bmatrix} \qquad (10.2.8)$$

which is an upper triangular matrix of order n. Similarly, a *lower triangular matrix* is a square matrix whose elements above the principal diagonal are all zero, and of the form:

$$A = \begin{bmatrix} a_{11} & 0 & 0 & ... & 0 \\ a_{21} & a_{22} & 0 & ... & 0 \\ ... & ... & ... & ... & \\ a_{n1} & a_{n2} & & ... & a_{nn} \end{bmatrix} \qquad (10.2.9)$$

which is a lower triangular matrix of order n.

10.2.8 Null or Zero Matrix

An $m \times n$ matrix with all the elements zero is called a *null or zero matrix* and is denoted by 0. To indicate the number of rows and columns of a zero matrix, one writes 0_{mn} e.g.,

$$A = \begin{bmatrix} 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \ldots & 0 \end{bmatrix} \tag{10.2.10}$$

10.3 Algebraic Operations on Matrices

10.3.1 Equality of Matrices

Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are considered equal if they have the same order, that is, they have the same number of rows and columns, and the corresponding entries are equal; thus,

$$a_{ij} = b_{ij}; \quad \text{for all } i, j \tag{10.3.1}$$

Example 10.3.1

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} x & y \\ w & z \end{bmatrix}$

Then, $A = B$ iff,

$$\begin{aligned} a &= x \\ b &= y \\ c &= w \\ d &= z \end{aligned}$$

Example 10.3.2

Solve the equation, $\begin{bmatrix} x+y & 3u+v \\ x-y & 2u-v \end{bmatrix} = \begin{bmatrix} 8 & 7 \\ 4 & 3 \end{bmatrix}$ using the properties of equality of matrices.

Solution

Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are equal iff, $a_{ij} = b_{ij}$ for all i, j. Thus, we have the following pairs of equations:

$$\left.\begin{aligned} x + y = 8 \\ x - y = 4 \end{aligned}\right\} \qquad (i)$$

$$\left.\begin{aligned} 3u + v = 7 \\ 2u - v = 3 \end{aligned}\right\} \qquad (ii)$$

Solving the two pairs of equation simultaneously, we have;

$$\begin{aligned} & x = 6,\ y = 2 \\ \text{and} \quad & u = 2,\ v = 1 \end{aligned}$$

10.3.2 Sum and Difference of Matrices

We define sum and difference for *compatible matrices* i.e. matrices of the same size or order. Thus, the sum or addition of two matrices of $A = [a_{ij}]$ and $B = [b_{ij}]$ of the same order is a matrix $C = [c_{ij}]$ of the same order with entries c_{ij} equal to the sums of the corresponding entries a_{ij} and b_{ij} of the matrices A and B; i.e.,

$$a_{ij} + b_{ij} = c_{ij}; \quad \text{for all } i, j$$

We denote the sum of A and B by $A + B$. Thus,

$$A + B = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \dots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \dots & a_{2n} + b_{2n} \\ \dots\dots\dots & \dots\dots\dots & \dots & \dots\dots\dots \\ a_{m1} + b_{m1} & a_{m2} + b_{m2} & \dots & a_{mn} + b_{mn} \end{bmatrix} \qquad (10.3.2)$$

The difference of two matrices is defined thus;

$$A - B = \begin{bmatrix} a_{11} - b_{11} & a_{12} - b_{12} & \dots & a_{1n} - b_{1n} \\ a_{21} - b_{21} & a_{22} - b_{22} & \dots & a_{2n} - b_{2n} \\ \dots\dots\dots & \dots\dots\dots & \dots & \dots\dots\dots \\ a_{m1} - b_{m1} & a_{m2} - b_{m2} & \dots & a_{mn} - b_{mn} \end{bmatrix} \qquad (10.3.3)$$

Example 10.3.3

Find the sum and difference of the following matrices:

$$A = \begin{bmatrix} 2 & 4 & 1 \\ 3 & 9 & 5 \end{bmatrix} \text{ and } B = \begin{bmatrix} 3 & 0 & 4 \\ -1 & 2 & 7 \end{bmatrix}$$

Solution

$$A + B = \begin{bmatrix} 2+3 & 4+0 & 1+4 \\ 3-1 & 9+2 & 5+7 \end{bmatrix} = \begin{bmatrix} 5 & 4 & 5 \\ 2 & 11 & 13 \end{bmatrix}$$

$$A - B = \begin{bmatrix} 2-3 & 4-0 & 1-4 \\ 3+1 & 9-2 & 5-7 \end{bmatrix} = \begin{bmatrix} -1 & 4 & -3 \\ 4 & 7 & -2 \end{bmatrix}$$

Properties of Matrix Addition

The following properties are observed when two or more matrices are added together:
Let $A = [a_{ij}]$, $B = [b_{ij}]$ and $C = [c_{ij}]$ be $m \times n$ matrices, then the following holds;

(i) $A + (B + C) = (A + B) + C$ *(associative law of addition)*

(ii) $A + B = B + A$ *(commutative law of addition)*

(iii) $A + 0 = 0 + A = A$ *(identity matrix, 0)*

(iv) $A + (-A) = 0$ *(additive inverse of A)*

(i) Associative Law of Addition: If three matrices of the same order, say, $A = [a_{ij}]$, $B = [b_{ij}]$ and $C = [c_{ij}]$ then the following is true:

$$A + (B + C) = (a_{ij}) + (b_{ij} + c_{ij}) = (a_{ij} + (b_{ij} + c_{ij}))$$

and

$$(A + B) + C = (a_{ij} + b_{ij}) + (c_{ij}) = ((a_{ij} + b_{ij}) + c_{ij})$$

Therefore, since,

$$(a_{ij}) + (b_{ij} + c_{ij}) = (a_{ij} + b_{ij}) + (c_{ij}); \text{ for all } i, j$$

it follows that,

$$A + (B + C) = (A + B) + C$$

and so, the associative law holds for the set of matrices of the same order with respect to addition.

(ii) Commutative Law of Addition: Let $A = [a_{ij}]$, and $B = [b_{ij}]$ be matrices of the same order, then the following is true:

$$A + B = (a_{ij} + b_{ij})$$

$$B + A = (b_{ij} + a_{ij})$$

Therefore, since,

$$a_{ij} + b_{ij} = b_{ij} + a_{ij}; \text{ for all } i, j,$$

it follows that,

$$A + B = B + A$$

hence the commutative law holds for the set of all matrices of the same order with respect to addition.

Example 10.3.4

Let A, B and C be any 2×2 matrices, show that they are associative and commutative with respect to addition where,

$$A = \begin{bmatrix} -1 & 4 \\ 5 & 7 \end{bmatrix}, \ B = \begin{bmatrix} 6 & 0 \\ -11 & -4 \end{bmatrix} \text{ and } C = \begin{bmatrix} 13 & -5 \\ 20 & 17 \end{bmatrix}$$

Solution

(i) If A, B and C are Associative then,

$$A + (B + C) = (A + B) + C$$

Thus,

$$(A + B) = \begin{bmatrix} -1 & 4 \\ 5 & 7 \end{bmatrix} + \begin{bmatrix} 6 & 0 \\ -11 & -4 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ -6 & 3 \end{bmatrix}$$

So,

$$(A + B) + C = \begin{bmatrix} 5 & 4 \\ -6 & 3 \end{bmatrix} + \begin{bmatrix} 13 & -5 \\ 20 & 17 \end{bmatrix} = \begin{bmatrix} 18 & -1 \\ 14 & 20 \end{bmatrix}$$

Similarly,

$$(B+C) = \begin{bmatrix} 6 & 0 \\ -11 & -4 \end{bmatrix} + \begin{bmatrix} 13 & -5 \\ 20 & 17 \end{bmatrix} = \begin{bmatrix} 19 & -5 \\ 9 & 13 \end{bmatrix}$$

Hence,

$$A+(B+C) = \begin{bmatrix} -1 & 4 \\ 5 & 7 \end{bmatrix} + \begin{bmatrix} 19 & -5 \\ 9 & 13 \end{bmatrix} = \begin{bmatrix} 18 & -1 \\ 14 & 20 \end{bmatrix}$$

Therefore, $A+(B+C) = (A+B)+C$ and hence, they are associative with respect to addition.

(ii) To show that they are commutative, we need to establish that $A+B = B+A$.
Thus,

$$A+B = \begin{bmatrix} -1+6 & 4+0 \\ 5-11 & 7-4 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ -6 & 3 \end{bmatrix}$$

$$B+A = \begin{bmatrix} 6-1 & 0+4 \\ -11+5 & -4+7 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ -6 & 3 \end{bmatrix}$$

Therefore, $A+B = B+A$ and hence, they are commutative with respect to addition.

10.3.3 Multiplication of a Matrix by a Scalar

The multiplication of a matrix $A = [a_{ij}]$ by a scalar α (or the product of a scalar and a matrix A) is a matrix whose entries are obtained by multiplying all the elements of A by the scalar α, such that,

$$A\alpha = \alpha A = \begin{bmatrix} \alpha a_{11} & \alpha a_{12} & \dots & \alpha a_{1n} \\ \alpha a_{21} & \alpha a_{22} & \dots & \alpha a_{2n} \\ \dots & \dots & \dots & \dots\dots \\ \alpha a_{m1} & \alpha a_{m2} & \dots & \alpha a_{mn} \end{bmatrix} \qquad (10.3.4)$$

Suppose A and B be matrices of the same order, and let α and β be scalars, then from the definition of the product of a scalar and a matrix, the following properties are satisfied:

(i) $(-1)A = -A$ (where the scalar, $\alpha = -1$)

(ii) $0A = 0$ (where the scaler $\alpha = 0$)

(iii) $(\alpha + \beta)A = \alpha A + \beta A$

(iv) $\alpha(A + B) = \alpha A + \alpha B$

(v) $\alpha(\beta A) = (\alpha\beta)A$

Example 10.3.5

Find the product of a 2×3 matrix: $A = \begin{bmatrix} 3 & -1 & -7 \\ 5 & -4 & 0 \end{bmatrix}$ and a scalar $\alpha = 3$.

Solution

$$\begin{aligned} \alpha A &= \begin{bmatrix} 3 \times 3 & 3 \times (-1) & 3 \times (-7) \\ 3 \times 5 & 3 \times (-4) & 3 \times 0 \end{bmatrix} \\ &= \begin{bmatrix} 9 & -3 & -21 \\ 15 & -12 & 0 \end{bmatrix} \end{aligned}$$

10.3.4 Multiplication of Matrices

If A and B are two matrices, we define a product, AB as; A multiplied by B. This relationship can only exist if the number of columns of A is equal to the number of rows of B. In other words, A cannot be multiplied by B if the number of columns of A is not equal to the number of rows of B.

Now, suppose,

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{in} \\ \dots & \dots & \dots & \dots \\ a_{i1} & a_{i2} & \dots & a_{in} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

and

$$B = \begin{bmatrix} b_{11} & ... & b_{ij} & ... & b_{1q} \\ b_{21} & ... & b_{2j} & ... & b_{2q} \\ ... & ... & ... & ... & ... \\ b_{p1} & ... & b_{pj} & ... & b_{pq} \end{bmatrix}$$

are matrices of order $m \times n$ and $p \times q$ respectively. If the number of columns of A is equal to the number of rows of B, i.e. $n = p$, then for these matrices is defined a *product matrix* C of order $m \times q$:

$$C = \begin{bmatrix} c_{11} & c_{12} & ... & c_{ij} & ... & c_{1q} \\ c_{21} & c_{22} & ... & c_{2j} & ... & c_{2q} \\ ... & ... & ... & ... & ... & ... \\ c_{i1} & c_{i2} & ... & c_{ij} & ... & c_{iq} \\ ... & ... & ... & ... & ... & ... \\ c_{m1} & c_{m2} & ... & c_{mj} & ... & c_{mq} \end{bmatrix}$$

where,

$$\begin{aligned} c_{ij} &= a_{i1}b_{1j} + a_{i2}b_{2j} + ... + a_{in}b_{nj} \\ &= \textstyle\sum_{k=1}^{n} a_{ik}b_{kj}, (i = 1, 2, ..., m; j = 1, 2, ..., q) \end{aligned} \tag{10.3.5}$$

From the definition above, it follows that to obtain the element in the ith row and jth column from a product of two matrices, multiply the elements of the ith row of the first matrix by the corresponding elements of the jth column of the second and add the products.

Consequently, if the product AB of matrices A and B is defined, we say that A and B are *conformable* (or *compatible*) matrices for multiplication. We then say that A is *post-multiplied* by B or that B is *pre-multiplied* by A. Otherwise, if the number of columns of A is not equal to the number of rows of B, then the product AB is not defined.

Example 10.3.5

Compute the products of the following matrices:

(i) Let $A = \begin{bmatrix} 2 \\ -7 \\ 5 \end{bmatrix}$; $\quad B = [6 \quad 1 \quad 0 \quad -9]$. Now, to test whether

A and B are compatible for multiplication:

- first, count the number of columns of A; here A has 1 column;
- secondly, count the number of rows of B; here B has 1 row;
- thirdly, check whether the number of columns of A is equal to the number of rows of B: here $1 = 1$, i.e. the number of columns of A is equal to the rows of B, and hence they are conformable for multiplication.

Thus,

$$\begin{aligned} AB &= \begin{bmatrix} 2 \\ -7 \\ 5 \end{bmatrix} [6 \quad 1 \quad 0 \quad -9] \\ &= \begin{bmatrix} 2\times 6 & 2\times 1 & 2\times 0 & 2\times(-9) \\ -7\times 6 & -7\times 1 & -7\times 0 & -7\times -9 \\ 5\times 6 & 5\times 1 & 5\times 0 & 5\times -9 \end{bmatrix} \\ &= \begin{bmatrix} 12 & 2 & 0 & -18 \\ -42 & -7 & 0 & 63 \\ 30 & 5 & 0 & -45 \end{bmatrix} \end{aligned}$$

which is a 3×4 matrix.

(ii) Let $A = \begin{bmatrix} -1 & 6 \\ 4 & 7 \end{bmatrix}$; $\quad B = \begin{bmatrix} 9 & -3 \\ 2 & 5 \end{bmatrix}$
Number of column of A= number of rows of $B = 2$

$$\begin{aligned} \therefore \qquad AB &= \begin{bmatrix} -1 & 6 \\ 4 & 7 \end{bmatrix} \begin{bmatrix} 9 & -3 \\ 2 & 5 \end{bmatrix} \\ &= \begin{bmatrix} -1\times 9 + 6\times 2 & -1\times -3 + 6\times 5 \\ 4\times 9 + 7\times 2 & 4\times -3 + 7\times 5 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 33 \\ 50 & 23 \end{bmatrix} \end{aligned}$$

(iii) Let $A = \begin{bmatrix} 3 & 4 \\ -1 & -5 \end{bmatrix}$; $\quad B = \begin{bmatrix} 7 & 2 & -3 \\ 0 & -1 & 6 \end{bmatrix}$

Number of column of A = Number of row of $B = 2$

$$AB = \begin{bmatrix} 3 & 4 \\ -1 & -5 \end{bmatrix} \begin{bmatrix} 7 & 2 & -3 \\ 0 & -1 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 3\times 7 + 4\times 0 & 3\times 2 + 4\times -1 & 3\times -3 + 4\times 6 \\ -1\times 7 + -5\times 0 & -1\times 2 + -5\times -1 & -1\times -3 + -5\times 6 \end{bmatrix}$$

$$= \begin{bmatrix} 21 & 2 & 15 \\ -7 & 3 & -27 \end{bmatrix}$$

(iv) $A = \begin{bmatrix} 0 & 9 & -1 \\ -5 & 4 & 2 \end{bmatrix}$; $\quad B = \begin{bmatrix} -2 \\ 1 \\ 7 \end{bmatrix}$

No of column of A = No of row of $B = 3$.

$$\begin{aligned} AB &= \begin{bmatrix} 0 & 9 & -1 \\ -5 & 4 & 2 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 7 \end{bmatrix} \\ &= \begin{bmatrix} 0\times -2 + 9\times 1 + -1\times 7 \\ -5\times -2 + 4\times 1 + 2\times 7 \end{bmatrix} = \begin{bmatrix} 2 \\ 28 \end{bmatrix} \end{aligned}$$

(v) $A = \begin{bmatrix} -1 & 0 & 3 \\ 2 & 4 & -5 \end{bmatrix}$; $\quad B = \begin{bmatrix} 0 & -1 \\ 4 & 5 \\ -7 & 2 \end{bmatrix}$

No of column of A = No of row of $B = 3$

$$\begin{aligned} AB &= \begin{bmatrix} -1 & 0 & 3 \\ 2 & 4 & -5 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 4 & 5 \\ -7 & 2 \end{bmatrix} \\ &= \begin{bmatrix} -1\times 0 + 0\times 4 + 3\times -7 & -1\times -1 + 0\times 5 + 3\times 2 \\ 2\times 0 + 4\times 4 + -5\times -7 & 2\times -1 + 4\times 5 + -5\times 2 \end{bmatrix} \\ &= \begin{bmatrix} -21 & 7 \\ 51 & 8 \end{bmatrix}. \end{aligned}$$

Example 10.3.6

Let $A = [3 \quad -4 \quad 7]$ and $B = \begin{bmatrix} 2 \\ -1 \\ 8 \end{bmatrix}$. Show that the product of A and B i.e. row matrix and a column matrix is a scalar.

Solution

$$
\begin{aligned}
AB = [3 \quad -4 \quad 7]\begin{bmatrix} 2 \\ -1 \\ 8 \end{bmatrix} &= [3 \times 2 + -4 \times -1 + 7 \times 8] \\
&= 66
\end{aligned}
$$

Example 10.3.7

Find all the values of x and y such that;

$$
\begin{bmatrix} 4 & 2x & 3 \\ -1 & 2 & 0 \end{bmatrix}\begin{bmatrix} 1 & 5 \\ 3 & 0 \\ -6 & 3y \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 5 & -5 \end{bmatrix}
$$

Solution

$$
\begin{aligned}
\begin{bmatrix} 4 & 2x & 3 \\ -1 & 2 & 0 \end{bmatrix}\begin{bmatrix} 1 & 5 \\ 3 & 0 \\ -6 & 3y \end{bmatrix} &= \begin{bmatrix} 4 + 6x - 18 & 20 + 0 + 9y \\ -1 + 6 + 0 & -5 + 0 + 0 \end{bmatrix} \\
&= \begin{bmatrix} 6x - 14 & 20 + 9y \\ 5 & -5 \end{bmatrix}
\end{aligned}
$$

$$
\text{i.e.,} \quad \begin{bmatrix} 6x - 14 & 20 + 9y \\ 5 & -5 \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 5 & -5 \end{bmatrix}
$$

Since two matrices A and B are equal iff $a_{ij} = b_{ij}$; for all i, j; then we have,

$$
\begin{aligned}
6x - 14 &= -2 \\
20 + 9y &= 2
\end{aligned}
$$

Now, solving for x and y we obtain, $x = 2$ and $y = -2$.

Properties of Matrix Multiplication

Let A, B and C be matrices with the operations; multiplication and addition defined and let α be a scalar, then the following properties are satisfied:

(i) $A(BC) = (AB)C$ *(associative law)*

(ii) $\alpha(AB) = (\alpha A)B$ *(scalar multiplication)*

(iii) $(A + B)C = AC + BC$ *(right distributive law)*

(iv) $C(A + B) = CA + CB$ *(left distributive law)*

(v) $IA = AI = A$ *(identity element, I)*

(i) Associative Law for Multiplication

Theorem 10.3.1. *Let $A = [a_{ij}]_{m,n}$, $B = [b_{jk}]_{n,p}$ and $C = [C_{kl}]_{p,q}$ be matrices such that, the product $(AB)C$ and $A(BC)$ are defined and are $m \times q$ matrices. Then the associative law of matrix multiplication holds, i.e. $(AB)C = A(BC)$*

Proof.

$$AB = [\textstyle\sum_{j=1}^{n} a_{ij}b_{jk}]_{m,p}$$

Thus ,

$$\begin{aligned}(AB)C &= \left[\textstyle\sum_{k=1}^{p} \left(\sum_{j=1}^{n} a_{ij}b_{jk}\right) C_{kl}\right]_{m,q} \\ &= \left[\textstyle\sum_{k=1}^{p} \sum_{j=1}^{n} a_{ij}b_{jk}C_{kl}\right]_{m,q} \end{aligned} \tag{10.3.6}$$

Also,

$$BC = [\textstyle\sum_{k=1}^{p} b_{jk}C_{kl}]_{n,q}$$

i.e.

$$A(BC) = \left[\textstyle\sum_{j=1}^{n} a_{ij} \left(\sum_{k=1}^{p} b_{jk}c_{kl}\right)\right]_{m,q} \tag{10.3.7}$$

$$= \left[\textstyle\sum_{j=1}^{n} \sum_{k=1}^{p} a_{ij}b_{jk}C_{kl}\right]_{m,q}$$

Since they are finite sums, equations (10.3.6) and (10.3.7) are equal, which completes the proof.
Hence $(AB)C = A(BC)$. □

Example 10.3.8

Let $A = \begin{bmatrix} 1 & -3 \\ 5 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 0 & 4 \\ 1 & -2 & -7 \end{bmatrix}$ and $C = \begin{bmatrix} -1 & 3 \\ 0 & 4 \\ 2 & -5 \end{bmatrix}$. Show that A, B and C are associative over multiplication.

Solution

$$\begin{aligned} AB &= \begin{bmatrix} 1 & -3 \\ 5 & 4 \end{bmatrix} \begin{bmatrix} 3 & 0 & 4 \\ 1 & -2 & -7 \end{bmatrix} \\ &= \begin{bmatrix} 3-3 & 0+6 & 4+21 \\ 15+4 & 0-8 & 20-28 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 6 & 25 \\ 19 & -8 & -8 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} (AB)C &= \begin{bmatrix} 0 & 6 & 25 \\ 19 & -8 & -8 \end{bmatrix} \begin{bmatrix} -1 & 3 \\ 0 & 4 \\ 2 & -5 \end{bmatrix} \\ &= \begin{bmatrix} 0+0+50 & 0+24-125 \\ -19+0-16 & 57-32+40 \end{bmatrix} \\ &= \begin{bmatrix} 50 & -101 \\ -35 & 65 \end{bmatrix} \end{aligned}$$

$$
\begin{aligned}
BC &= \begin{bmatrix} 3 & 0 & 4 \\ 1 & -2 & -7 \end{bmatrix} \begin{bmatrix} -1 & 3 \\ 0 & 4 \\ 2 & -5 \end{bmatrix} \\
&= \begin{bmatrix} -3+0+8 & 9+0-20 \\ -1+0-14 & 3-8+35 \end{bmatrix} \\
&= \begin{bmatrix} 5 & -11 \\ -15 & 30 \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
A(BC) &= \begin{bmatrix} 1 & -3 \\ 5 & 4 \end{bmatrix} \begin{bmatrix} 5 & -11 \\ -15 & 30 \end{bmatrix} \\
&= \begin{bmatrix} 5+45 & -11-90 \\ 25-60 & -55+120 \end{bmatrix} \\
&= \begin{bmatrix} 50 & -101 \\ -35 & 65 \end{bmatrix}
\end{aligned}
$$

Therefore, $(AB)C = A(BC)$, and hence A, B and C are associative over multiplication.

(ii) Distributive Laws

Theorem 10.3.2. *Let $A = [a_{ij}]_{m,n}$, $B = [b_{ij}]_{m,n}$, and $C = [C_{jk}]_{n,p}$ be matrices such that AC, BC and $A + B$ are defined. Then the following distributive law for matrix multiplication holds:*

$$(A + B)C = AC + BC$$

Similarly, if $D = [d_{jk}]_{q,m}$ is a matrix such that DA and DB are defined, then the following distributive law for matrix multiplication holds:

$$D(A + B) = DA + DB$$

Proof.

$$\begin{aligned}
A + B &= [a_{ij} + b_{ij}]_{m,n} \\
\text{Thus, } (A+B)C &= \left[(a_{ij} + b_{ij}) + \left(\sum_{j=1}^{n} c_{jk}\right)\right]_{m,p} \\
&= \left[\sum_{j=1}^{n} a_{ij}C_{jk} + \sum_{j=1}^{n} b_{ij}C_{jk}\right]_{m,p} \\
&= \left[\sum_{j=1}^{n} a_{ij}C_{jk}\right]_{m,p} + \left[\sum_{j=1}^{n} b_{ij}C_{jk}\right]_{m,p} \\
&= AC + BC
\end{aligned} \tag{10.3.8}$$

Which completes the proof of the first distributive law. The proof of the second law is analogous and is left as an exercise. □

Example 10.3.9

Let $A = \begin{bmatrix} -1 & 0 & 4 \\ 2 & -3 & 5 \\ 1 & 3 & -7 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 1 & 3 \\ 0 & -2 & 7 \\ 2 & 1 & -6 \end{bmatrix}$ and

$C = \begin{bmatrix} -1 & 1 \\ 0 & -2 \\ 2 & -7 \end{bmatrix}$. Show that $(A + B)C = AC + BC$ i.e. satisfy the distributive property for matrix multiplication.

Solution

$$\begin{aligned}
A + B &= \begin{bmatrix} -1+4 & 0+1 & 4+3 \\ 2+0 & -3+(-2) & 5+7 \\ 1+2 & 3+1 & -7+(-6) \end{bmatrix} \\
&= \begin{bmatrix} 3 & 1 & 7 \\ 2 & -5 & 12 \\ 3 & 4 & -13 \end{bmatrix}
\end{aligned}$$

$$
\begin{aligned}
(A+B)C &= \begin{bmatrix} 3 & 1 & 7 \\ 2 & -5 & 12 \\ 3 & 4 & -13 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 0 & -2 \\ 2 & -7 \end{bmatrix} \\
&= \begin{bmatrix} -3+0+14 & 3-2-49 \\ -2+0+24 & 2+10+8 \\ -3+0-26 & 3-8+91 \end{bmatrix} \\
&= \begin{bmatrix} 11 & -48 \\ 22 & -72 \\ -29 & 86 \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
AC &= \begin{bmatrix} -1 & 0 & 4 \\ 2 & -3 & 5 \\ 1 & 3 & -7 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 0 & -2 \\ 2 & -7 \end{bmatrix} \\
&= \begin{bmatrix} 1+0+8 & -1-28 \\ -2+0+10 & 2+6-35 \\ -1+0+(-14) & 1-6+49 \end{bmatrix} \\
&= \begin{bmatrix} 9 & -29 \\ 8 & -27 \\ -15 & 44 \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
BC &= \begin{bmatrix} 4 & 1 & 3 \\ 0 & -2 & 7 \\ 2 & 1 & -6 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 0 & -2 \\ 2 & -7 \end{bmatrix} \\
&= \begin{bmatrix} -4+6 & 4-2-21 \\ 14 & 4-49 \\ -2-12 & 2-2+42 \end{bmatrix} \\
&= \begin{bmatrix} 2 & -19 \\ 14 & -45 \\ -14 & 42 \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
AC+BC &= \begin{bmatrix} 9 & -29 \\ 8 & -27 \\ -15 & 44 \end{bmatrix} + \begin{bmatrix} 2 & -19 \\ 14 & -45 \\ -14 & 42 \end{bmatrix} \\
&= \begin{bmatrix} 11 & -48 \\ 22 & -72 \\ -29 & 86 \end{bmatrix}
\end{aligned}
$$

Therefore, $(A+B)C = AC+BC$, and hence A, B and C are distributive over multiplication.

(iii) Identity Element

Theorem 10.3.3. *Let M_n be the set of all square matrices of order n, and denote I_n as the identity element of this set over multiplication. Now, suppose $A \epsilon M_n$, then,*

$$I_n A = A = A.I_n \tag{10.3.9}$$

Proof. Previously, we defined δ_{ij} as,

$$\delta_{ij} = \begin{cases} 1, & \text{if} \quad i = j \\ 0, & \text{if} \quad i \neq j \end{cases}$$

By using this symbol, we can define the $n \times n$ square matrix I_n thus;

$$I_n = [\delta_{ij}]_{n,n}$$

It follows that, this matrix has only 1's on the principal diagonal. To show that I_n is truly the identity matrix of order n, we need to show that if,

$$\begin{aligned} A &= [a_{ij}]_{n,n} \epsilon M_n \\ \text{then,} \qquad AI_n &= A = I_n A \end{aligned}$$

It follows that,

$$\begin{aligned} AI_n &= \left[\sum_{j=1}^{n} a_{ij}\delta_{jk} \right]_{n,n} \\ &= [a_{ij}]_{n,n} \\ &= A \end{aligned}$$

and

$$\begin{aligned} I_n A &= \left[\sum_{j=1}^{n} \delta_{ij} a_{jk} \right]_{n,n} \\ &= [a_{ik}]_{n,n} \\ &= A \end{aligned}$$

which completes the proof. □

Example 10.3.10

Let $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and A a square matrix of order 2, i.e. $A = \begin{bmatrix} W & X \\ Y & Z \end{bmatrix}$, then,

$$I_2A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} W & X \\ Y & Z \end{bmatrix} = \begin{bmatrix} W+0 & x+0 \\ 0+y & 0+Z \end{bmatrix}$$

$$= \begin{bmatrix} W & X \\ Y & Z \end{bmatrix} = A$$

and

$$AI_2 = \begin{bmatrix} W & X \\ Y & Z \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} W+0 & 0+X \\ Y+0 & 0+Z \end{bmatrix}$$

$$= \begin{bmatrix} W & X \\ Y & Z \end{bmatrix} = A$$

(iv) Commutative Law for Multiplication

The product of two matrices is generally *non-commutative*, i.e. if A and B are matrices that are conformable for multiplication for the product AB; it is common that $AB \neq BA$ in general. Indeed BA may not even exist except A and B are square matrices. However, if $AB = BA$, then we say that A and B *commute* or are *commutative*.

Nevertheless, as observed in equation 10.3.9, the identity matrix I commutes with any square matrix M_n.

Example 10.3.11

Show or otherwise whether the following matrices A and B commutes, where
$A = \begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & -1 \\ 3 & 5 \end{bmatrix}$.

Solution

$$A = \begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix}; \quad B = \begin{bmatrix} -2 & -1 \\ 3 & 5 \end{bmatrix}$$

Thus,

$$AB = \begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} -2 & -1 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} -2+9 & -1+15 \\ -8+15 & -4+25 \end{bmatrix}$$
$$= \begin{bmatrix} 7 & 14 \\ 7 & 21 \end{bmatrix}$$

Similarly,

$$BA = \begin{bmatrix} -2 & -1 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} -2-4 & -6-5 \\ 3+20 & 9+25 \end{bmatrix}$$
$$= \begin{bmatrix} -6 & -11 \\ 23 & 34 \end{bmatrix}$$

$\therefore \quad AB \neq BA$ and hence A and B are not commutative.

Example 10.3.12

Find all the 2×2 matrices which commute with the matrix: $A = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$.

Solution

Let B be a 2×2, i.e. $B = \begin{bmatrix} x & y \\ u & v \end{bmatrix}$.
Thus, we shall find the elements of B such that $AB = BA$ i.e.,

$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x & y \\ u & v \end{bmatrix} = \begin{bmatrix} x & y \\ u & v \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

On multiplication, we obtain;

$$\begin{bmatrix} x+2u & y+2v \\ 0+0 & 0+0 \end{bmatrix} = \begin{bmatrix} x+0 & 2x+0 \\ u+0 & 2u+0 \end{bmatrix}$$

Thus,by equating corresponding elements, we have;

$$\begin{aligned} x+2u &= x & (i) \\ y+2v &= 2x & (ii) \\ 0 &= u & (iii) \\ 0 &= 2u & (iv) \end{aligned}$$

From (i) and (ii), $x = 0$ is a trivial solution i.e., from (ii) with $x = 0$,

$$y = -2v$$

$\therefore$ all matrices of the form:

$$B = \begin{bmatrix} 0 & -2v \\ 0 & v \end{bmatrix}$$

where v is any number, commutes with $A = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$ such that,

$$\begin{aligned} \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & -2v \\ 0 & v \end{bmatrix} &= \begin{bmatrix} 0 & -2v \\ 0 & v \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

(v) The Cancelation Law for Multiplication

The *cancelation law* is not defined in matrices unlike in algebra of real numbers where the law states that if;

$$\begin{aligned} ab &= 0 \\ \text{implies that either} \quad a &= 0 \text{ or } b = 0 \end{aligned}$$

and if $ab = bc$, it implies that $a = c$; where a, b and c are real numbers.

Consider the matrices,

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & -4 \\ -1 & 2 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} 2 & -4 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

But $A \neq 0$ and $B \neq 0$.
Similarly, consider the following matrices:

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}, \; B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \; C = \begin{bmatrix} 3 & 3 \\ 0 & 0 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 6 & 6 \end{bmatrix}$$

and

$$AC = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 3 & 3 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 6 & 6 \end{bmatrix}$$

$\therefore \quad AB = AC$, but $B \neq C$

10.4 Elementary Operations on Matrices

The following elementary row (column) operations may be applied to matrices:

(i) Interchanging two rows (or columns) denoted by $R_{ij}(C_{ij})$,

(ii) Multiplying all elements of one row (column) by the same non-zero scalar, α denoted by $R_{ij}(\alpha)[C_{ij}(\alpha)]$,

(iii) Adding multiples of the elements of a row (column) to elements of another row (column).

Definition 20 *Two matrices are termed Equivalent if one is obtained from the other by carrying out a finite number of elementary operations. Such matrices are not generally speaking equal, but, as may be proved later, they have the same rank [10.4.3]. In other words, a matrix is said to be row - equivalent to another matrix B denoted by $A \sim B$, if B can be obtained from A by elementary row operation . Similarly, A is said to be column-equivalent to B, if B can be obtained from A by elementary column operations. Note* *that the relation of equivalence among matrices is an equivalence relation satisfying the reflexive, symmetric and transitive laws in Section 5.5.1*

10.4.1 Elementary Row Operations on Matrices

We shall, for simplicity write the elementary row operation as (e.r.o). Now, suppose we have a matrix $A = \begin{bmatrix} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{bmatrix}$, we can perform the following row operations: $R_1 \leftrightarrow R_3, R_2 \rightarrow 2R_2$, $R_3 \rightarrow -\frac{1}{2}R_1 + R_3$ on the rows of the above matrix A respectively as follows;

(i) $R_1 \leftrightarrow R_3 \Rightarrow \begin{bmatrix} 4 & 9 & 2 \\ 3 & 5 & 7 \\ 8 & 1 & 6 \end{bmatrix}$ i.e. interchange of R_1 and R_3

(ii) $R_2 \rightarrow 2R_2 \Rightarrow \begin{bmatrix} 8 & 1 & 6 \\ 6 & 10 & 14 \\ 4 & 9 & 2 \end{bmatrix}$ i.e. multiply R_2 by 2

(iii) $R_3 \rightarrow -\frac{1}{2}R_1 + R_3 \Rightarrow \begin{bmatrix} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 0 & 8\frac{1}{2} & -1 \end{bmatrix}$

i.e. multiply R_1 by $-\frac{1}{2}$ and add to R_3

For simplicity however, we shall use the symbol ε to denote an e.r.o, such that $\varepsilon(A)$ means the matrix that results from performing the e.r.o ε on the matrix A. Please note that the order of e.r.o.s. in a sequence matters, as the operations are not commutative.

If there exists two matrices A type, say $m \times n$. We say that A is row-equivalent to B denoted by $A \sim B$, if and only if there is a sequence of e.r.o.s $\varepsilon_1, \varepsilon_2, ..., \varepsilon_n$ that transforms A into B, such that;

$$B = \varepsilon_1\varepsilon_2\varepsilon_3...\varepsilon_n(A) \tag{10.4.1}$$

10.4.2 Echelon Matrices and Reduced Echelon Matrices

A matrix $A = [a_{ij}]$ is called an *echelon matrix* or is said to be in echelon form if the number of zeros preceding the first non-zero element or entry of a row increase row by row until only zero rows remain i.e., an echelon matrix must satisfy the following two properties:

(i) If $R_1, R_2, ..., R_i$ are the non-zero rows (from top to bottom), then the leading entry (i.e. the first non-zero entry in any row) of R_{i+1} occurs strictly farther right than the leading entry of R_i; $(i = 1, 2, 3, ..., i - 1)$.

(ii) The zero rows if any, occur at the bottom

Example 10.4.1

Let A and B be given respectively as;

(i) $$A = \begin{bmatrix} \boxed{2} & -4 & 3 \\ 0 & \boxed{-2} & 1 \\ 0 & 0 & \boxed{5} \\ 0 & 0 & 0 \end{bmatrix}$$

(ii) $$B = \begin{bmatrix} \boxed{1} & 2 & 4 & 3 \\ 0 & \boxed{-5} & 0 & 2 \\ 0 & 0 & 0 & \boxed{1} \end{bmatrix}$$

Thus, A and B are echelon matrices with the boxed entries as the distinguished elements or leading entries.

Special Echelon Matrices
They include;

(i) A null or zero matrix, $\underline{0}$

(ii) An $n \times 1$ or column matrix i.e, $\begin{bmatrix} 1 \\ 2 \\ 3 \\ \vdots \\ 0 \end{bmatrix}$

Canonical or Normal form of Row-Reduced Echelon Matrices

A row reduced echelon matrix is said to be in the canonical or normal form if

(i) In each non-zero row, the leading entry is 1

(ii) In each column that contains the leading entry of a row, all other entries are zero;

(iii) The leading entry in the first row is the $(1, 1)$-position; otherwise perform a type (i) e.r.o. to bring a non-zero entry to the top of the column.

Example 10.4.2

Let A and B be two matrices, such that;

$$A = \begin{bmatrix} 1 & 5 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}; \; B = \begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Observe that the leading entries are 1's and are the only non-zero entry in their respective column; so they are all in the canonical or normal form.

10.4.3 Rank of a Matrix

Rows or columns of a matrix A, are said to be linearly dependent, if there exists scalars, not all of them zero, such that the linear combination of the row or columns is equal to zero, otherwise, the rows or columns are said to be linearly independent.

By rank of a matrix A, denoted by r, we mean the number of maximally independent rows or columns. In other words, the row rank = the column rank = the number of non-zero rows in echelon form.

Thus, to find the rank of a matrix, we first reduce it to row - reduced echelon form by using elementary row - operations and the rank becomes the number of non-zero rows contained in the echelon matrix.

Example 10.4.3

Find the rank of the matrix $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 3 & 1 \end{bmatrix}$ by transforming the matrix A into a row - reduced echelon form and hence obtain the row canonical or normal form.

Solution

To find the rank, we first reduce the matrix to echelon form by using elementary row operations as follows:

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 3 & 1 \end{bmatrix} \xrightarrow{R_2 \to -R_1 + R_2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix} \xrightarrow{R_3 \to -3R_2 + R_3} \begin{bmatrix} \boxed{1} & 0 & 1 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & \boxed{1} \end{bmatrix}$$

Alternatively,

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 3 & 1 \end{bmatrix} \xrightarrow{R_2 \to -R_1 + R_2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix} \xrightarrow{R_3 \to \frac{1}{3} R_3} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & \frac{1}{3} \end{bmatrix}$$

$$\xrightarrow{R_3 \to -R_2 + R_3} \begin{bmatrix} \boxed{1} & 0 & 1 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & \boxed{1/3} \end{bmatrix} = E(A \sim E)$$

Observe that we obtained two echelon forms from only one matrix above, the boxed entries are the leading entries or distinguished elements. Furthermore, the number of the non-zero rows in the echelon form is 3

$\therefore$ the rank =3

Now, to reduce the row-reduced echelon form to row - canonical or normal form, we have the following:

$$\xrightarrow{R_1 \to -R_3+R_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ OR } \xrightarrow{R_1 \to -3R_3+R_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$$

and

$$\xrightarrow{R_3 \to 3R_3} \begin{bmatrix} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & \boxed{1} \end{bmatrix}$$

which is the required row-canonical or normal form.

Remark 24 *A given matrix has different echelon matrices, i.e. the echelon matrix is not unique from the above example. But the basic property that the rank is the same is that they contain the same number of non-zero rows.*

Example 10.4.4

Find the rank of the following matrices:

(i) $A = \begin{bmatrix} 0 & 3 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}$

(ii) $B = \begin{bmatrix} 2 & -4 & 3 & 1 & 0 \\ 1 & -2 & 1 & -4 & 2 \\ 0 & 1 & -1 & 3 & 1 \\ 4 & -7 & 4 & -4 & 5 \end{bmatrix}$

and hence obtain their normal form.

Solution

(i) $A = \begin{bmatrix} 0 & 3 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}$ Since the entry at the (1,1)-position is zero, we interchange the rows by;

$$\begin{bmatrix} 0 & 3 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix} \xrightarrow{R_3 \leftrightarrow R_1} \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 1 \\ 0 & 3 & 1 \end{bmatrix} \xrightarrow{R_2 \to -R_1+R_2} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & -1 \\ 0 & 3 & 1 \end{bmatrix}$$

$$R_2 \leftrightarrow R_3 \longrightarrow \begin{bmatrix} \boxed{1} & 1 & 2 \\ 0 & \boxed{3} & 1 \\ 0 & 0 & \boxed{-1} \end{bmatrix} \xrightarrow[R_3 \to -R_3]{R_2 \to \frac{1}{3}R} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & \frac{1}{3} \\ 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_1 \to -R_2 + R_1} \begin{bmatrix} 1 & 0 & \frac{5}{3} \\ 0 & 1 & \frac{1}{3} \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[R_2 \to -\frac{1}{3}R_3 + R_2]{R_1 \to -\frac{5}{3}R_3 + R_1} \begin{bmatrix} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & \boxed{1} \end{bmatrix}$$

which is the normal form.

$\therefore$ the rank $= 3$

(ii)

$$B = \begin{bmatrix} 2 & -4 & 3 & 1 & 0 \\ 1 & -2 & 1 & -4 & 2 \\ 0 & 1 & -1 & 3 & 1 \\ 4 & -7 & 4 & -4 & 5 \end{bmatrix} \xrightarrow[R_4 \to -4R_1 + R_4]{R_1 \to \frac{1}{2}R_1,\ R_2 \to -R_1 + R_2}$$

$$= \begin{bmatrix} 1 & -2 & \frac{3}{2} & \frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} & -\frac{9}{2} & 2 \\ 0 & 1 & -1 & 3 & 1 \\ 0 & 1 & -2 & -6 & 5 \end{bmatrix} \xrightarrow[R_3 \to -R_4 + R_3]{R_2 \leftrightarrow R_4}$$

$$= \begin{bmatrix} 1 & -2 & \frac{3}{2} & \frac{1}{2} & 0 \\ 0 & 1 & -2 & -6 & 5 \\ 0 & 0 & 1 & 9 & -4 \\ 0 & 0 & -\frac{1}{2} & -\frac{9}{2} & 2 \end{bmatrix} \xrightarrow{R_4 \to \frac{1}{2}R_3 + R_4}$$

$$= \begin{bmatrix} \boxed{1} & -2 & \frac{3}{2} & \frac{1}{2} & 0 \\ 0 & \boxed{1} & -2 & -6 & 5 \\ 0 & 0 & \boxed{1} & 9 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

is the echelon form, hence the rank = 3.

Now, to obtain the canonical form, we have;

$$\begin{bmatrix} 1 & -2 & \frac{3}{2} & \frac{1}{2} & 0 \\ 0 & 1 & -2 & -6 & 5 \\ 0 & 0 & 1 & 9 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[R_2 \to 2R_3 + R_2]{R_1 \to 2R_2 + R_1}$$

$$= \begin{bmatrix} 1 & 0 & -\frac{5}{2} & -\frac{23}{2} & 10 \\ 0 & 1 & 0 & 12 & -3 \\ 0 & 0 & 1 & 9 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 \to \frac{5}{2}R_3 + R_1}$$

$$= \begin{bmatrix} \boxed{1} & 0 & 0 & \frac{23}{2} & 0 \\ 0 & \boxed{1} & 0 & 12 & -3 \\ 0 & 0 & \boxed{1} & 9 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

which is the required canonical form.

Remark 25 *We can, in a similar manner, reduce any matrix to a column-reduced echelon form by using elementary column operations and the column rank of the matrix will be equal to the number of non-zero columns in the column-reduced echelon matrix.*

10.5 Elementary Matrices

Let E be a matrix corresponding to the e.r.o., ε, then E is the matrix $\varepsilon(I_n)$ which results from performing ε on I_n and is called elementary matrix corresponding to ε.

Example 10.5.1

Perform the following e.r.o. on I_3 respectively:
(i) $R_1 \to -R_3 + R_1$, (ii) $R_2 \to 2R_1 + R_2$,
(iii) $R_3 \to -\frac{1}{2}R_2 + R_3$.

Solution

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{(i)}\ R_1 \rightarrow -R_3 + R_1 \Rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E_i$$

$$\text{(ii)}\ R_2 \rightarrow 2R_1 + R_2 \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E_{ii}$$

$$\text{(iii)}\ R_3 \rightarrow -\tfrac{1}{2}R_2 + R_3 \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{1}{2} & 1 \end{bmatrix} = E_{iii}$$

Theorem 10.5.1. *Let ε be any e.r.o. (applicable to n-rowed matrices) and let E be the $n \times n$ elementary matrix corresponding to ε, then for every matrix A,*

$$\varepsilon(A) = EA \qquad (10.4.2)$$

We shall establish the proof of this theorem by the following example.

Example 10.5.2

Let $A = \begin{bmatrix} 0 & 3 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$, by performing the following e.r.o.s:
(i) $R_1 \leftrightarrow R_3$; (ii) $R_2 \rightarrow -R_1 + R_2$; (iii) $R_3 \rightarrow -3R_2 + R_3$, on A consequently, show that;

$$\varepsilon_3\varepsilon_2\varepsilon_1 A = E_3 E_2 E_1 A$$

Solution

$$A = \begin{bmatrix} 0 & 3 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \xrightarrow[R_1 \leftrightarrow R_3]{\varepsilon_1} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 3 & 1 \end{bmatrix}$$

the elementary matrix E_1, associated with ε, is;

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = E_1,$$

and

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 3 & 1 \end{bmatrix} \xrightarrow[R_2 \to -R_1 + R_2]{\varepsilon_2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix}$$

$$\xrightarrow[R_3 \to -3R_2 + R_3]{\varepsilon_3} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and the elementary matrices corresponding to e.r.o.s ε_2 and ε_3 are;

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 \to -R_1 + R_2} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E_2,$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_3 \to -3R_2 + R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix} = E_3$$

$$\therefore \qquad \varepsilon_3\varepsilon_2\varepsilon_1 A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and so,}$$

$$E_3E_2E_1A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 3 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore \qquad \varepsilon_3\varepsilon_2\varepsilon_1 A = E_3E_2E_1A \tag{10.4.3}$$

Note that $E_1E_2 \neq E_2E_1$, i.e. the order is strictly maintained accordingly for ε and E.
Generally, for any $n \times n$ matrix, say A, a finite elementary row operation on A will transform A into an $n \times n$ identity matrix I_n i.e.

$$\varepsilon_n ... \varepsilon_3\varepsilon_2\varepsilon_1 A = I_n. \tag{10.4.4}$$

Note

For each e.r.o., ε; we specify as detailed below, an e.r.o. called the inverse of ε and denoted by ε^{-1}:

e.r.o.(ε)	**Inverse of e.r.o.** (ε^{-1})
$R_i \leftrightarrow R_j$	$R_i \leftrightarrow R_j$
$R_i \rightarrow kR_i$; $(k \neq 0)$	$R_i \rightarrow (\frac{1}{k})R_i$; $(k \neq 0)$
$R_i \rightarrow kR_j + R_i$	$R_i \rightarrow -kR_j + R_i$

Consequently, the inverses of the elementary matrix E, denoted by E^{-1} can be obtained by changing the sign of the entries below the main diagonal (i.e. the negatives of the multipliers, k_{ij}). Notice that k_{ij} corresponds to $R_i \rightarrow kR_j + R_i$. For example the inverse of the above elementary matrices E_1, E_2 and E_3 of Example 10.5.2 is as follows:
E_1 is a trivial case since the inverse E_1^{-1} is simply the reverse of the operation i.e. $E_1^{-1} = E_1$:

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \quad E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and}$$

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}; \quad E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix}$$

10.6 Triangular Matrices

A triangular matrix was briefly discussed in Section 10.2.7 as consisting of upper triangular and lower triangular matrix. It is a square matrix whose entries below or above the principal diagonal is all zero.

For example,

$$U = \begin{bmatrix} t_{11} & t_{12} & ... & t_{in} \\ 0 & t_{22} & ... & t_{2n} \\ ... & ... & ... & ... \\ 0 & 0 & ... & t_{nn} \end{bmatrix}$$

where $t_{ij} = 0$ for $i > j$ is an *upper triangular matrix*. Similarly,

$$L = \begin{bmatrix} t_{11} & 0 & ... & 0 \\ t_{21} & t_{22} & ... & 0 \\ ... & ... & ... & ... \\ t_{n1} & t_{n2} & ... & t_{nn} \end{bmatrix}$$

where $t_{ij} = 0$ for $j > i$ is a *lower triangular matrix*. In other words a diagonal matrix is a special case of both an upper and lower triangular matrix.

Theorem 10.6.1. *The sum and product of triangular matrices of the same order and the same structure (i.e. upper only or lower only) are also triangular matrices of the same order and structure.*

We shall illustrate the proof of this theorem by the use of example.

Example 10.6.1

Find the sum and product of the following upper triangular matrices:

$$U_1 = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -5 & 7 \\ 0 & 0 & -1 \end{bmatrix}; \; U_2 = \begin{bmatrix} 4 & -1 & -2 \\ 0 & 2 & -3 \\ 0 & 0 & 4 \end{bmatrix}$$

$$U_1 + U_2 = \begin{bmatrix} 1+4 & 2-1 & 2-2 \\ 0+0 & -5+2 & 7-3 \\ 0+0 & 0+0 & -1+4 \end{bmatrix} = \begin{bmatrix} 5 & 1 & 1 \\ 0 & -3 & 4 \\ 0 & 0 & 3 \end{bmatrix}$$

which is the required triangular matrix.

$$U_1U_2 = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -5 & 7 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 4 & -1 & -2 \\ 0 & 2 & -3 \\ 0 & 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -1+4 & -2-6+12 \\ 0 & -10 & 15+28 \\ 0 & 0 & -4 \end{bmatrix} = \begin{bmatrix} 4 & 3 & 4 \\ 0 & -10 & 43 \\ 0 & 0 & -4 \end{bmatrix}$$

which is the required triangular matrix.

10.6.1 L U Decomposition

In this section, our objective is to show how to decompose a given n-square matrix into a product of a lower triangular matrix (L) and an upper triangular matrix (U) by performing a finite number of e.r.o.s on the given matrix under certain conditions.

If A is an $n \times n$ matrix, and suppose E_1, E_2, and E_3 are the elementary matrices corresponding to the e.r.o.s. performed on A, then;

$$E_3E_2E_1A = U \tag{10.6.1}$$

where U is an upper triangular matrix. However, since the inverses of the elementary matrices exists then,

$$(E_1^{-1}E_2^{-1}E_3^{-1})(E_3E_2E_1A) = E_1^{-1}E_2^{-1}E_3^{-1}U \tag{10.6.2}$$

which is equivalent to,

$$A = E_1^{-1}E_2^{-1}E_3^{-1}U = LU \tag{10.6.3}$$

where,

$$L = E_1^{-1}E_2^{-1}E_3^{-1} \tag{10.6.4}$$

which is the lower triangular matrix.

Example 10.6.2

Using the LU decomposition method, show that the matrix $A = LU$ given that,

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \\ -1 & 4 & -3 \end{bmatrix}.$$

Solution

The e.r.o.s. we are going to perform are:

(i) $R_2 \to -2R_1 + R_2$

(ii) $R_3 \to R_1 + R_3$

(iii) $R_3 \to 2R_2 + R_3$

Thus, the result of the above e.r.o.s on A is,

$$\varepsilon_3\varepsilon_2\varepsilon_1 A = E_3E_2E_1A = U = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -5 \\ 0 & 0 & -10 \end{bmatrix}$$

So that the elementary matrices corresponding to the above e.r.o.s are:

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \quad \text{i.e.,} \quad E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}; \quad \text{i.e.,} \quad E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}; \quad \text{i.e.,} \quad E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

Thus,

$$\begin{aligned} L = E_1^{-1}E_2^{-1}E_3^{-1} &= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -2 & 1 \end{bmatrix} \end{aligned}$$

$$\therefore \qquad A = LU$$

Hence,

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \\ -1 & 4 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -5 \\ 0 & 0 & -10 \end{bmatrix}$$

Note that the LU decomposition can be decomposed further, so that U will have 1's on its main diagonal just as L has.

We can do this by dividing out of U a diagonal matrix D, made up entirely of the pivots $d_1, d_2, ..., d_n$ such that

$$U = \begin{bmatrix} d_1 & 0 & ... & 0 \\ 0 & d_2 & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & d_n \end{bmatrix} \begin{bmatrix} 1 & u_{12}/d_1 & U_{13}/d_1 & ... & U_{1n}/d_1 \\ 0 & 1 & u_{23}/d_2 & ... & u_{2n}/d_2 \\ ... & ... & ... & ... & ... \\ 0 & 0 & ... & ... & 1 \end{bmatrix} \tag{10.6.5}$$

For convenient, we denote this new triangular matrix by the same letter U, so that whenever we see LDU, it is understood that U has 1's on the main diagonal.

Example 10.6.3

Using our previous Example 10.6.2, show that $A = LDU$.

Solution

$$U_1 = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -5 \\ 0 & 0 & -10 \end{bmatrix}; \text{ i.e } d_1 = 1, d_2 = -3 \text{ and } d_3 = -10$$

So that,

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -10 \end{bmatrix}$$

Thus,

$$U_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -10 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & \frac{5}{3} \\ 0 & 0 & 1 \end{bmatrix} = DU$$

$$\therefore \qquad A = LDU \tag{10.6.6}$$

Theorem 10.6.2. *Any square matrix:*

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

with a non-zero principal-diagonal may be represented as a product of two triangular matrices of different structure (lower and upper); this expansion will be unique if the diagonal elements of one of the triangular matrices are fixed beforehand (say by putting them equal to 1).

We shall however omit the proof of this theorem in our discuss here, but will indicate a method for finding the entries or elements of a desired triangular matrices. Suppose,

$$A = LU \tag{10.6.7}$$

where

$$L = [b_{ij}], \quad b_{ij} = 0; \;\; \text{when } j > i \tag{10.6.8}$$

is a lower triangular matrix of order n,,

$$U = [c_{ij}]; \; c_{ij} = 0 \;\; \text{when} \;\; i > j \tag{10.6.9}$$

is an upper triangular matrix of order n. Thus multiplying the two matrices L and U as in (10.6.7), we obtain;

$$A = LU = \sum_{k=1}^{n} b_{ik}C_{kj} = a_{ij}; \; (i, j = 1, 2, ..., n) \tag{10.6.10}$$

Applying conditions (10.6.8) and (10.6.9), equation (10.6.10) becomes;

$$\sum_{k=1}^{j} b_{ik}c_{kj} = a_{ij}; \;\; \text{for } i \geq j (j = 1, 2, ...n) \tag{10.6.11}$$

and

$$\sum_{k=1}^{1} b_{ik}C_{kj} = a_{ij}; \quad \text{for } i < j(i = 1, 2, ..., n-1) \qquad (10.6.12)$$

Observe that because of their peculiar structure, the eqns. (10.6.11) and (10.6.12) are readily solved to within the diagonal elements b_{ii} and c_{ii}. However, for the sake of definiteness, we shall put $c_{ii} = 1, (i = 1, 2, ..., n)$.

Example 10.6.4

Represent the matrix $A = \begin{bmatrix} 2 & -3 & 4 \\ 1 & 2 & -3 \\ -1 & -2 & 5 \end{bmatrix}$ as a product of two triangular matrices L and U.

Solution

We seek L and U in the form:

$$L = \begin{bmatrix} t_{11} & 0 & 0 \\ t_{21} & t_{22} & 0 \\ t_{31} & t_{32} & t_{33} \end{bmatrix}; \; U = \begin{bmatrix} 1 & r_{12} & r_{13} \\ 0 & 1 & r_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

such that,

$$A = LU$$

By applying equations (10.6.11) for L when $i \geq j$ and (10.6.12) to obtain U when $i < j$; and putting $c_{ii} = 1$, for $i = (1, 2, 3)$ we have,

$$\begin{bmatrix} 2 & -3 & 4 \\ 1 & 2 & -3 \\ -1 & -2 & 5 \end{bmatrix} = \begin{bmatrix} t_{11} & t_{11}r_{12} & t_{11}r_{13} \\ t_{21} & t_{21}r_{12} + t_{22} & t_{21}r_{13} + t_{22}r_{23} \\ t_{31} & t_{31}r_{12} + t_{32} & t_{31}r_{13} + t_{32}r_{23} + t_{33} \end{bmatrix}$$

where,

$$\begin{aligned} &t_{11} = 2, && t_{11}r_{12} = -3, && t_{11}r_{13} = 4 \\ &t_{21} = 1, && t_{21}r_{12} + t_{22} = 2, && t_{21}r_{13} + t_{22}r_{23} = -3 \\ &t_{31} = -1, && t_{31}r_{12} + t_{32} = -2, && t_{31}r_{13} + t_{32}r_{23} + t_{33} = 5 \end{aligned}$$

and solving the system, we obtain;

$$\begin{array}{lll} t_{11} = 2, & t_{21} = 1 & t_{31} = -1 \\ t_{22} = \frac{7}{2}, & t_{32} = -\frac{7}{2}, & t_{33} = 2 \\ r_{12} = \frac{-3}{2}, & r_{23} = \frac{-10}{7}, & r_{13} = 2 \end{array}$$

Thus,

$$L = \begin{bmatrix} 2 & 0 & 0 \\ 1 & \frac{7}{2} & 0 \\ -1 & \frac{-7}{2} & 2 \end{bmatrix}$$

and

$$U = \begin{bmatrix} 1 & \frac{-3}{2} & 2 \\ 0 & 1 & \frac{-10}{7} \\ 0 & 0 & 1 \end{bmatrix}$$

The reader is advised to check by finding the product of L and U such that $LU = A$.

10.6.2 Row Exchanges and Permutation Matrices

Recall that type (i) elementary row operation is the interchanging of two rows (10.4) denoted by $R_i \leftrightarrow R_j$. However, instead of manually exchanging two rows whenever it is necessary, we need to express this operation in matrix terms; such that the required *permutation matrix* denoted by p is that matrix that produces the row exchange desired. In other words, a permutation matrix is a special type of elementary matrix obtained from a type (i) elementary row operation. It is obtained by applying the permutation to the identity matrix. I_n as we did in elementary matrix. We shall illustrate with examples as follows:

Example 10.6.5

Put the following 3×3 matrix A, in echelon form using permutation matrices P, given that $A = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & -3 \\ 1 & 2 & 3 \end{bmatrix}$

Solution

$$P_{13}A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & -3 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & -3 \\ 0 & 1 & 2 \end{bmatrix}$$

$$P_{23}A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & -3 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & -3 \end{bmatrix}$$

which is the required echelon form. However, the product of any two permutation matrix is another permutation matrix, so that the two row exchanges above can be performed by multiplying the two permutation matrix together to give;

$$P_{23}P_{13} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = P$$

So that,

$$PA = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & -3 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} \boxed{1} & 2 & 3 \\ 0 & \boxed{1} & 2 \\ 0 & 0 & \boxed{\text{-3}} \end{bmatrix}$$

which gives the same result as the first. Note the order of operation of the permutation matrix as the product is not also commutative.

10.7 Trace of a Matrix

Let $A = [a_{ij}]_{n,n}$ be a square matrix of order n, and let a_{ii} be the diagonal elements, then the trace of a matrix A is the sum of the principal diagonal elements i.e.,

$$\text{trace } (A) = \sum_{i=1}^{n} a_{ii} \quad (i = 1, 2, ..., n) \tag{10.6.13}$$

Example 10.7.1

If $A = \begin{bmatrix} 4 & -2 & 3 \\ 1 & 2 & 5 \\ 0 & 6 & -1 \end{bmatrix}$.

Thus, trace $(A) = \sum_{i=1}^{3} a_{ii} = 4 + 2 - 1 = 5$

10.8 Transpose of a Matrix

Let A be an $m \times n$ matrix, then the *transpose* of A denoted by A' or $A^\top$ is an $n \times m$ matrix such that the element a_{ij} of $A^\top$ is equal to the element a_{ji} of A for all i and j, i.e. if

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

to obtain the transpose of A, we replace the rows by the columns to get,

$$A^\top = \begin{bmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & \dots & a_{m2} \\ \dots & \dots & \dots & \dots \\ a_{1n} & a_{2n} & \dots & a_{mn} \end{bmatrix} \tag{10.6.14}$$

which is of order $n \times m$. In particular, the transpose of the row vector,

$$a = [a_1 a_2 ... a_n]$$

is the column vector,

$$a^\top = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \tag{10.6.15}$$

Example 10.8.1

If

$$A = \begin{bmatrix} 1 & -4 & 0 \\ 2 & 5 & -2 \\ 3 & 7 & 6 \\ -8 & 2 & 1 \end{bmatrix} \Rightarrow A^\top = \begin{bmatrix} 1 & 2 & 3 & -8 \\ -4 & 5 & 7 & 2 \\ 0 & -2 & 6 & 1 \end{bmatrix}$$

10.8.1 Properties of Matrix Transposition

Let A and B be matrices, then;

(i) $(A+B)^T = A^\top + B^\top$

(ii) $A^{\top\top} = (A^\top)^\top = A$

(iii) $(\alpha A)^\top = \alpha A^\top; \quad (\alpha \text{ a scalar})$

(iv) $(AB)^\top = B^\top A^\top$

We shall illustrate the proof of these properties by examples, however, if the reader wishes to prove them, he can make reference to the properties of matrix multiplication in Section 10.3.4 for guidance.

Example 10.8.2

(i) $A = \begin{bmatrix} 4 & 1 & -5 \\ -2 & 3 & -1 \end{bmatrix}$; $B = \begin{bmatrix} -1 & 3 & 6 \\ 2 & 4 & -5 \end{bmatrix}$

Then, $(A+B)^\top = \begin{bmatrix} 3 & 4 & 1 \\ 0 & 7 & -6 \end{bmatrix}^\top = \begin{bmatrix} 3 & 0 \\ 4 & 7 \\ 1 & -6 \end{bmatrix}$

$$A^\top + B^\top = \begin{bmatrix} 4 & -2 \\ 1 & 3 \\ -5 & -1 \end{bmatrix} + \begin{bmatrix} -1 & 2 \\ 3 & 4 \\ 6 & -5 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 4 & 7 \\ 1 & -6 \end{bmatrix}$$

$$\therefore \qquad (A+B)^\top = A^\top + B^\top$$

(ii) If $A = \begin{bmatrix} 4 & 1 & -5 \\ -2 & 3 & -1 \end{bmatrix} \Rightarrow A^\top = \begin{bmatrix} 4 & -2 \\ 1 & 3 \\ -5 & -1 \end{bmatrix}$

and $(A^\top)^\top = \begin{bmatrix} 4 & -2 \\ 1 & 3 \\ -5 & -1 \end{bmatrix}^\top = \begin{bmatrix} 4 & 1 & -5 \\ -2 & 3 & -1 \end{bmatrix}$

$$\therefore \qquad (A^\top)^\top = A$$

(iii) Let $\alpha = -2$, and $A = \begin{bmatrix} 4 & 1 & -5 \\ -2 & 3 & -1 \end{bmatrix}$

then

$$\begin{aligned} (\alpha A)^\top &= \begin{bmatrix} 4 \times (-2) & 1 \times (-2) & -5 \times (-2) \\ -2 \times (-2) & 3 \times (-2) & -1 \times (-2) \end{bmatrix}^\top \\ &= \begin{bmatrix} -8 & 4 \\ -2 & -6 \\ 10 & 2 \end{bmatrix} \end{aligned}$$

and $\alpha A^\top = -2 \begin{bmatrix} 4 & -2 \\ 1 & 3 \\ -5 & -1 \end{bmatrix} = \begin{bmatrix} -8 & 4 \\ -2 & -6 \\ 10 & 2 \end{bmatrix}$

$$\therefore \qquad (\alpha A)^\top = \alpha A^\top$$

(iv) Let $A = \begin{bmatrix} 4 & 1 & -5 \\ -2 & 3 & -1 \end{bmatrix}; \quad B = \begin{bmatrix} -1 & 2 \\ 3 & 4 \\ 6 & -5 \end{bmatrix}$

then $(AB)^\top = \begin{bmatrix} -31 & 37 \\ 5 & 13 \end{bmatrix}^\top = \begin{bmatrix} -31 & 5 \\ 37 & 13 \end{bmatrix}$

and

$$B^\top A^\top = \begin{bmatrix} -1 & 3 & 6 \\ 2 & 4 & -5 \end{bmatrix} \begin{bmatrix} 4 & -2 \\ 1 & 3 \\ -5 & -1 \end{bmatrix} = \begin{bmatrix} -31 & 5 \\ 37 & 13 \end{bmatrix}$$

$$\therefore \qquad (AB)^\top = B^\top A^\top$$

10.8.2 Symmetric and Skew-Symmetric Matrices

A matrix A is said to be *Symmetric* if and only if the entry a_{ij} of $A^\top$ is equal to the entry a_{ij} of A for all i, j i.e. if $A^\top = A$ then $\forall i, j$; $a_{ij} = a_{ji}$, for the given matrix $A = [a_{ij}]_{n,n}$.

It follows that;

(i) A symmetric matrix is square ($m = n$);

(ii) The elements symmetric about the principal diagonal are equal.

Example 10.8.3

$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$ is a symmetric matrix since $A^\top = A$;

or more explicitly, if $A = \begin{bmatrix} -1 & 2 & 3 \\ 2 & 5 & -4 \\ 3 & -4 & 7 \end{bmatrix}$

Observe that $a_{ij} = a_{ji}$ and hence $A^\top = A$. So, A is symmetric.

Similarly a matrix $A = [a_{ij}]_{n,n}$ (i.e. square) is said to be *Skew - Symmetric*, if and only if the entry a_{ij} of $A^\top$ is equal to the negative of the entry a_{ij} of A for all i, j, i.e. if

$$A^\top = -A$$

then for all i, j, $a_{ij} = -a_{ji}$ for the given matrix A. It is also necessary that the principal diagonal elements be equal to zero i.e. $a_{ii} = 0$

Example 10.8.4

$$A = \begin{bmatrix} 0 & -X & -Y \\ X & 0 & -Z \\ Y & Z & 0 \end{bmatrix}$$

is a Skew-Symmetric matrix since $A^\top = -A$.

Theorem 10.8.1. *Any square matrix say $A = [a_{ij}]_{n,n}$, can be written as the sum of a symmetric matrix $B = [b_{ij}]$ and a Skew-Symmetric matrix $C = [c_{ij}]$.*

Proof. Suppose,

$$A = \frac{1}{2}(A + A^\top) + \frac{1}{2}(A - A^\top)$$

where,

$$B = \frac{1}{2}(A + A^\top), \quad C = \frac{1}{2}(A - A^\top)$$

We want to show that,

(i) $A = B + C$

(ii) B is Symmetric

(iii) C is Skew-Symmetric

(i) $A = B + c$ follows from:

$$\begin{aligned} B + C &= \frac{1}{2}(A + A^\top) + \frac{1}{2}(A - A^\top) \\ &= \frac{1}{2}A + \frac{1}{2}A^\top + \frac{1}{2}A - \frac{1}{2}A^\top \\ &= \frac{1}{2}A + \frac{1}{2}A \\ &= A \end{aligned}$$

(ii) To show that B is Symmetric:

$$\begin{aligned} B^\top &= \frac{1}{2}(A + A^\top)^\top \\ &= \frac{1}{2}A^\top + A^{\top\top} \\ &= \frac{1}{2}A^\top + A \\ &= \frac{1}{2}(A + A^\top) \\ &= B \end{aligned}$$

(iii) To show that C is Skew-Symmetric:

$$\begin{aligned} C^\top &= \frac{1}{2}(A - A^\top)^\top \\ &= \frac{1}{2}A^\top - \frac{1}{2}A^{\top\top} \\ &= \frac{1}{2}A^\top - \frac{1}{2}A \\ &= -\frac{1}{2}(A - A^\top) \\ &= -C \end{aligned}$$

which completes the proof

$\square$

Example 10.8.5

Verify the above proof by the following matrices: (i) $A = \begin{bmatrix} 4 & 3 \\ 9 & -1 \end{bmatrix}$
(ii) $A = \begin{bmatrix} 3 & 5 & -7 \\ 2 & 1 & 0 \\ 4 & -3 & -8 \end{bmatrix}$

Solution

(i) Recall that $B = \frac{1}{2}(A + A^\top)$ from the proof, hence,

$$\begin{aligned} B = \tfrac{1}{2}(A + A^\top) &= \tfrac{1}{2}\left\{\begin{bmatrix} 4 & 3 \\ 9 & -1 \end{bmatrix} + \begin{bmatrix} 4 & 9 \\ 3 & -1 \end{bmatrix}\right\} \\ &= \tfrac{1}{2}\left\{\begin{bmatrix} 8 & 12 \\ 12 & -2 \end{bmatrix}\right\} \\ &= \begin{bmatrix} 4 & 6 \\ 6 & -1 \end{bmatrix} \end{aligned}$$

$\therefore$ B is symmetric.
Thus,

$$C = \frac{1}{2}(A - A^\top) = \frac{1}{2}\left\{\begin{bmatrix} 0 & -6 \\ 6 & 0 \end{bmatrix}\right\} = \begin{bmatrix} 0 & -3 \\ 3 & 0 \end{bmatrix}$$

$\therefore \quad C$ is Skew-symmetric.
Hence,

$$B + C = \begin{bmatrix} 4 & 6 \\ 6 & -1 \end{bmatrix} + \begin{bmatrix} 0 & -3 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ 9 & -1 \end{bmatrix} = A$$

(ii) Given that, $A = \begin{bmatrix} 3 & 5 & -7 \\ 2 & 1 & 0 \\ 4 & -3 & -8 \end{bmatrix}$

Thus,

$$\begin{aligned} B &= \tfrac{1}{2}(A + A^\top) \\ &= \tfrac{1}{2}\left\{ \begin{bmatrix} 3 & 5 & -7 \\ 2 & 1 & 0 \\ 4 & -3 & -8 \end{bmatrix} + \begin{bmatrix} 3 & 2 & 4 \\ 5 & 1 & -3 \\ -7 & 0 & -8 \end{bmatrix} \right\} \\ &= \tfrac{1}{2}\left\{ \begin{bmatrix} 6 & 7 & -3 \\ 7 & 2 & -3 \\ -3 & -3 & -16 \end{bmatrix} \right\} \\ &= \begin{bmatrix} 3 & \frac{7}{2} & -\frac{3}{2} \\ \frac{7}{2} & 1 & -\frac{3}{2} \\ -\frac{3}{2} & -\frac{3}{2} & -8 \end{bmatrix} \end{aligned}$$

$\therefore \quad B$ is Symmetric.

Similarly,

$$\begin{aligned}
C &= \tfrac{1}{2}(A - A^{\top}) \\
&= \tfrac{1}{2}\left\{\begin{bmatrix} 3 & 5 & -7 \\ 2 & 1 & 0 \\ 4 & -3 & -8 \end{bmatrix} - \begin{bmatrix} 3 & 2 & 4 \\ 5 & 1 & -3 \\ -7 & 0 & -8 \end{bmatrix}\right\} \\
&= \tfrac{1}{2}\left\{\begin{bmatrix} 0 & 3 & -11 \\ -3 & 0 & 3 \\ 11 & -3 & 0 \end{bmatrix}\right\} \\
&= \begin{bmatrix} 0 & \frac{3}{2} & \frac{-11}{2} \\ \frac{-3}{2} & 0 & \frac{3}{2} \\ \frac{11}{2} & \frac{-3}{2} & 0 \end{bmatrix}
\end{aligned}$$

$\therefore$ C is a Skew Symmetric matrix.
It follows that,

$$\begin{aligned}
B + C &= \begin{bmatrix} 3 & \frac{7}{2} & \frac{-3}{2} \\ \frac{7}{2} & 1 & \frac{-3}{2} \\ \frac{-3}{2} & \frac{-3}{2} & -8 \end{bmatrix} + \begin{bmatrix} 0 & \frac{3}{2} & \frac{-11}{2} \\ \frac{-3}{2} & 0 & \frac{3}{2} \\ \frac{11}{2} & \frac{-3}{2} & 0 \end{bmatrix} \\
&= \begin{bmatrix} 3 & 5 & -7 \\ 2 & 1 & 0 \\ 4 & -3 & -8 \end{bmatrix} \\
&= A
\end{aligned}$$

This verifies the proof.

Example 10.8.6

Given that $A = \begin{bmatrix} 4 & 3x & 1 \\ x+2 & \cos\theta & 6z-3 \\ -y & 2 & -\sin\theta \end{bmatrix}$

$B = \begin{bmatrix} 3 & 4x+3 & \frac{5}{6}y+1 \\ 2 & 0 & \frac{2}{5} \\ 5 & 3z & 4 \end{bmatrix}$ and $\theta = 90°$

Show that $A+B=C$, where C is a Symmetric matrix, hence or otherwise find the values of x, y and z.

Solution

$$A+B=\begin{bmatrix} 7 & 7x+3 & \frac{5}{6}y+2 \\ x+4 & 0 & 6z-\frac{13}{5} \\ 5-y & 3z+2 & 3 \end{bmatrix}=C$$

to show that C is symmetric, then $a_{ij}=a_{ji}$; for all i,j.
Thus,

$$\begin{aligned} 7x+3 &= x+4 && (i) \\ \frac{5}{6}y+2 &= 5-y && (ii) \\ 6z-\frac{13}{5} &= 3z+2 && (iii) \end{aligned}$$

Now, solving the systems, we obtain;

$$x=\frac{1}{6},\quad y=\frac{18}{11}\quad \text{and}\quad z=\frac{23}{15}$$

Therefore,

$$C=\begin{bmatrix} 7 & \frac{25}{6} & \frac{37}{11} \\ \frac{25}{6} & 0 & \frac{33}{5} \\ \frac{37}{11} & \frac{33}{5} & 3 \end{bmatrix}$$

is the desired matrix.

10.9 Computation of Determinants

For a square matrix $A=[a_{ij}]$ of order n, we assign a specific number called the determinant of A, denoted by $\det(A)$, $\triangle(A)$ or $|A|$ and is of the form:

$$\begin{vmatrix} a_{11} & a_{12} & \ldots\ldots\ldots & a_{1n} \\ a_{21} & a_{22} & \ldots\ldots\ldots & a_{2n} \\ \ldots. & \ldots. & \ldots\ldots\ldots & \ldots. \\ a_{n1} & a_{n2} & \ldots\ldots\ldots & a_{nn} \end{vmatrix}$$

Note that this $n \times n$ array of numbers (real, rational or complex) enclosed by vertical bars or straight lines, called the determinants of order n, do not mean absolute value neither a matrix; but denotes the number that the determinant functions assigns to the enclosed array of numbers, i.e. the enclosed square matrix. In other words, the determinant is a scalar (real, rational or complex)) associated with any square matrix.

For $n = 1$ and $n = 2$, we have these definitions;

(i) $n = 1, \quad \det(A) = |a_{11}| = a_{11}$
which implies that the determinant of a scalar is a scalar.

(ii) $n = 2$,

$$\begin{aligned} \det(A) &= \begin{vmatrix} a_{11} & & a_{12} \\ & \times & \\ a_{21} & & a_{22} \end{vmatrix} \\ &= aa_{11}a_{22} - a_{12}a_{21} \end{aligned}$$

Example 10.9.1

Evaluate the determinant of:

(i) $A = \begin{bmatrix} -2 & 3 \\ 4 & 5 \end{bmatrix}$

(ii) $A = \begin{bmatrix} 1 & -5 \\ 7 & 4 \end{bmatrix}$

Solution

(i)

$$\begin{aligned} \det(A) &= \begin{vmatrix} -2 & 3 \\ 4 & 5 \end{vmatrix} \\ &= -2(5) - 4 \times 3 \\ &= -10 - 12 = -22 \end{aligned}$$

(ii) $\det(A) = \begin{vmatrix} 1 & -5 \\ 7 & 4 \end{vmatrix} = 1 \times 4 - (-5) \times 7 = 4 + 35 = 39$

10.9.1 Determinants of a 3×3 Matrix

We can write down the determinants of a matrix $A = [a_{ij}]$ of order 3 as

$$\begin{aligned}
|A| &= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \\
&= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\
&= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) \\
&\quad + a_{13}(a_{21}a_{32} - a_{22}a_{31}) \\
&= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} \\
&\quad + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} \\
&= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\
&\quad - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}
\end{aligned}$$

Observe that this is performed inductively, as it is written in terms of determinants of order two. Observe that on the right are six products i.e. 3! permutations, (Section 7.2). Each involves one factor a_{1i} from the first row, a_{2j} from the second row and a_{3k} from the third. Each column is also represented in every product so that $\det(A)$ has the sum of all the signed products of the form $\pm a_{1i}a_{2j}a_{3k}$, where i, j, k is some permutation of the column indices $1, 2, 3$. Note that among the six possible permutations, the three even permutations appear in products with prefix plus sign $(+)$, while the odd permutations are associated with a minus sign (-).

Experience has shown that the same principles apply to the solution of n equations in n unknowns. Each row appears once and only once in each term of $|A|$, which means that A is a linear and homogeneous function of the entries $a_{i1}, a_{i2}, ..., a_{in}$ in the i-th row of A. Collecting the coefficients of each a_{ij}, we obtain an expression:

$$\begin{aligned} |A| &= A_{i1}a_{i1} + A_{i2}a_{i2} + ... + A_{in}a_{in} \\ &= \sum_{j=1}^{n} A_{ij}a_{ij} = \sum_{j=1}^{n} a_{ij}A_{ij} \end{aligned} \tag{10.9.1}$$

where the coefficients A_{ij} of a_{ij} is referred to as the *cofactor* of a_{ij}, it is a polynomial in the entries of the remaining rows of A. Since, each term of $|A|$ involves each row and each column only once, the cofactor A_{ij} can involve neither the ith row nor the j-th column. It contains only entries from the *minor* or *sub matrix* M_{ij} which is the matrix obtained from A by crossing out the i-th row and j-th column. In other words, the cofactor A_{ij} of a_{ij} is the signed minor; viz:

$$A_{ij} = (-1)^{i+j}|M_{ij}| \tag{10.9.2}$$

Example 10.9.2

Evaluate the determinant of; $A = \begin{bmatrix} -11 & 2 & 2 \\ -4 & 1 & 0 \\ 6 & -1 & -1 \end{bmatrix}$

Solution

$$\begin{aligned} |A| &= \begin{vmatrix} -11 & 2 & 2 \\ -4 & 1 & 0 \\ 6 & -1 & -1 \end{vmatrix} \\ &= -11\begin{vmatrix} 1 & 0 \\ -1 & -1 \end{vmatrix} - 2\begin{vmatrix} -4 & 0 \\ 6 & -1 \end{vmatrix} + 2\begin{vmatrix} -4 & 1 \\ 6 & -1 \end{vmatrix} \\ &= -11(-1-0) - 2(4-0) + 2(4-6) \\ &= 11 - 8 + 8 - 12 = -1 \end{aligned}$$

Example 10.9.3

Let $A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{bmatrix}$; with the aid of the cofactors of A, evaluate the determinant of A.

Solution

$A_{ij} = (-1)^{i+j}|M_{ij}|$, thus;

$$|M_{11}| = \begin{vmatrix} -1 & 3 \\ 1 & 8 \end{vmatrix}$$

$$|M_{12}| = \begin{vmatrix} 2 & 3 \\ 4 & 8 \end{vmatrix}, \quad |M_{13}| = \begin{vmatrix} 2 & -1 \\ 4 & 1 \end{vmatrix}$$

So that,

$$A_{11} = (-1)^{1+1}\begin{vmatrix} -1 & 3 \\ 1 & 8 \end{vmatrix} = -8 - 3 = -12$$

$$A_{12} = (-1)^{1+2}\begin{vmatrix} 2 & 3 \\ 4 & 8 \end{vmatrix} = -(16 - 12) = -4$$

$$A_{13} = (-1)^{1+3}\begin{vmatrix} 2 & -1 \\ 4 & 1 \end{vmatrix} = 2 - (-4) = 6$$

Now, to find the $\det(A)$, we multiply each element of the first row of A by its cofactor and add to get;

$$\begin{aligned} \det(A) &= \sum_{j=1}^{3} a_{ij}A_{ij} \\ &= 1 \times A_{11} + 3 \times A_{12} + 2 \times A_{13} \\ &= -12 + 3(-4) + 2(6) \\ &= -12 - 12 + 12 \\ &= -12 \end{aligned}$$

Remark 26 *There is a simple check board arrangement for easy remembrance of the signs that corresponds to* $(-1)^{i+j}$ *and that can be applied to convert the minor of an element in a matrix to the cofactor of the element. The check board pattern is of the form:*

$$\begin{bmatrix} + & - & + & - & ...... \\ - & + & - & + & ...... \\ + & - & + & - & .... \\ ... & ... & ... & ... & ...... \end{bmatrix}$$

Observe that the determinant of any $n \times n$ *matrix can be computed using this pattern by first reducing it to* n *determinants of* $(n-1) \times (n-1)$ *matrices and so on until we arrive at the determinant of only* 2×2 *matrices whose computation is simple.*

10.9.2 Determinants of Higher Order $(n > 3)$

Elementary matrix transformations offer the most convenient and less cumbersome approach for computing the determinant of a matrix whose order is greater than 3. Suppose, for example, we have;

$$\triangle_n = \begin{bmatrix} a_{11} & a_{12} & ... & a_{in} \\ a_{21} & a_{22} & ... & a_{2n} \\ ... & ... & ... & ... \\ a_{n1} & a_{n2} & ... & a_{nn} \end{bmatrix}$$

Assuming that $a_{11} \neq 0$; dividing the first column by a_{11}, we obtain,

$$\triangle_n = a_{11} \begin{bmatrix} 1 & a_{12} & ... & a_{1n} \\ \frac{a_{21}}{a_{11}} & a_{22} & ... & a_{2n} \\ ... & ... & ... & ... \\ \frac{a_{n1}}{a_{11}} & a_{n2} & ... & a_{nn} \end{bmatrix}$$

Here, the value 1 obtained by dividing through by a_{11} now becomes the pivot element. Thus, subtracting from the elements of a_{ij} of the j-th column $(j \geq 2)$ the corresponding elements of the first column multiplied by a_{1j}, we have;

$$\triangle_n = a_{11} \begin{bmatrix} 1 & 0 & ... & 0 \\ \frac{a_{21}}{a_{11}} & a_{22}^{(1)} & & a_{2n}^{(1)} \\ ... & ... & ... & ... \\ \frac{a_{n1}}{a_{11}} & a_{n2}^{(1)} & ... & a_{nn}^{(1)} \end{bmatrix} = a_{11} \triangle_{n-1}$$

where,

$$\triangle_{n-1} = \begin{vmatrix} a_{22}^{(1)} & a_{23}^{(1)} & ... & a_{2n}^{(1)} \\ a_{32}^{(1)} & a_{33}^{(1)} & ... & a_{3n}^{(1)} \\ ... & ... & ... & ... \\ a_{n2}^{(1)} & & & a_{nn}^{(1)} \end{vmatrix}$$

and

$$a_{ij}^{(1)} = a_{ij} - \frac{a_{i1}a_{1j}}{a_{11}};\ (i, j = 2, 3, ...n)$$

Now, applying the same technique to the determinant $\triangle_{n-1}$ we finally obtain;

$$\triangle_n = a_{11}a_{22}^{(1)}...a_{nn}^{(n-1)},$$

where all the elements,

$$a_{ii}^{(i-1)} \neq 0;\ (i = 1, 2, ...n).$$

Note that if some intermediate determinant, $\triangle_{n-r}$ the upper left element (i.e. pivot element) $a_{r+1,r+1}^{(r)} = 0$, then it is necessary to interchange the rows or columns of the determinant $\triangle_{n-r}$ so that the pivot element we need is non-zero, which is always possible since the determinant, $\triangle \neq 0$. We must remember to take into consideration the change in the sign of $\triangle_{n-r}$, wherever, we interchange a row or column of the determinant.

For flexibility, we shall give a more general rule. Suppose the determinant $\triangle_n = \det[b_{ij}]$ is transformed so that $b_{pq} = 1$ (b_{pq} is the principal or pivot element), i.e.,

$$\bar{\triangle}_n = \begin{vmatrix} b_{11} & ... & b_{1q} & ... & b_{1j} & ... & b_{1n} \\ ... & ... & ... & ... & ... & ... & ... \\ b_{i1} & ... & b_{iq} & ... & b_{ij} & ... & b_{in} \\ ... & ... & ... & ... & ... & ... & ... \\ b_{p1} & ... & 1 & ... & b_{pj} & ... & b_{pn} \\ ... & ... & ... & ... & ... & ... & ... \\ b_{n1} & ... & b_{nq} & ... & b_{nj} & ... & b_{nn} \end{vmatrix}$$

Then,

$$\bar{\triangle}_n = (-1)^{p+q}\bar{\triangle}_{n-1} \tag{10.9.3}$$

where

$$\bar{\triangle}_{n-1} = \det[b_{ij}^1]$$

is determinant of the $(n-1)$th order obtained from $\bar{\triangle}_n$ by deleting the pth row and qth column with a subsequent transformation of the elements via the formula:

$$b_{ij}^{(1)} = b_{ij} - b_{iq}b_{pj}$$

Thus, each element $b_{ij}^{(1)}$ of the determinant $\bar{\triangle}_{n-1}$ is equal to the corresponding element b_{ij} of the determinant $\bar{\triangle}_n$ reduced by the product of its projections b_{iq} and b_{pj} by the deleted column and row of the original determinant. Anyway, we shall not be concerned with the proof of this proposition in our discussion, but shall illustrate by use of example to bring home our claim.

Example 10.9.4

Using elementary matrix transformation method, evaluate the determinants of a 5×5 matrix:

$$A = \begin{bmatrix} 3 & 1 & -1 & 2 & 1 \\ -2 & 3 & 1 & 4 & 3 \\ 1 & 4 & 2 & 3 & 1 \\ 5 & -2 & -3 & 5 & -1 \\ -1 & 1 & 2 & 3 & 2 \end{bmatrix}$$

Solution

Interchange column 1 and column 2 as follows:

$$\triangle_5 = -\begin{vmatrix} \boxed{1} & 3 & -1 & 2 & 1 \\ 3 & -2 & 1 & 4 & 3 \\ 4 & 1 & 2 & 3 & 1 \\ -2 & 5 & -3 & 5 & -1 \\ 1 & -1 & 2 & 3 & 2 \end{vmatrix}$$

Taking $a_{11} = 1$ for the principal or pivot element, we have the element $b_{ij} = a_{ij}^{(1)}$ of the $(5-1)$ determinant as follows:

$$\begin{aligned}
b_{11} &= a_{22}^{(1)} = a_{22} - a_{21}a_{12} = -2 - (3 \times 3) = -11 \\
b_{12} &= a_{23}^{(1)} = a_{23} - a_{21}a_{13} = 1 - (3 \times (-1)) = 4 \\
b_{13} &= a_{24}^{(1)} = a_{24} - a_{21}a_{14} = 4 - (3 \times 2) = -2 \\
b_{14} &= a_{25}^{(1)} = a_{25} - a_{21}a_{15} = 3 - (3 \times 1) = 0
\end{aligned}$$

$$\begin{aligned}
b_{21} &= a_{32}^{(1)} = a_{32} - a_{31}a_{12} = 1 - (4 \times 3) = -11 \\
b_{22} &= a_{33}^{(1)} = a_{33} - a_{31}a_{13} = 2 - (4 \times -1) = 6 \\
b_{23} &= a_{34}^{(1)} = a_{34} - a_{31}a_{14} = 3 - (4 \times 2) = -5 \\
b_{24} &= a_{35}^{(1)} = a_{35} - a_{31}a_{15} = 1 - (4 \times 1) = -3
\end{aligned}$$

$$\begin{aligned}
b_{31} &= a_{42}^{(1)} = a_{42} - a_{41}a_{12} = 5 - (-2 \times 3) = 11 \\
b_{32} &= a_{43}^{(1)} = a_{43} - a_{41}a_{13} = -3 - (-2 \times -1) = -5 \\
b_{33} &= a_{44}^{(1)} = a_{44} - a_{41}a_{14} = 5 - (-2 \times 2) = 9 \\
b_{34} &= a_{45}^{(1)} = a_{45} - a_{41}a_{15} = -1 - (-2 \times) = 1
\end{aligned}$$

$$\begin{aligned}
b_{41} &= a_{52}^{(1)} = a_{52} - a_{51}a_{12} = -1 - (1 \times 3) = -4 \\
b_{42} &= a_{53}^{(1)} = a_{53} - a_{51}a_{13} = 2 - (1 \times -1) = 3 \\
b_{43} &= a_{54}^{(1)} = a_{54} - a_{51}a_{14} = 3 - (1 \times 2) = 1 \\
b_{44} &= a_{55}^{(1)} = a_{55} - a_{51}a_{15} = 2 - (1 \times 1) = 1
\end{aligned}$$

$$\text{i.e.} \qquad \triangle_5 = -\begin{vmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{vmatrix} = -\begin{vmatrix} -11 & 4 & -2 & 0 \\ -11 & 6 & -5 & -3 \\ 11 & -5 & 9 & 1 \\ -4 & 3 & 1 & \boxed{1} \end{vmatrix}$$

Now, taking $b_{44} = 1$ for the pivot element and applying a similar transformation, we obtain;

$$\bar{\triangle}_4 = -(-1)^{4+4} \begin{vmatrix} -11 & 4 & -2 & 0 \\ -11 & 6 & -5 & -3 \\ 11 & -5 & 9 & 1 \\ -4 & 3 & 1 & \boxed{1} \end{vmatrix}$$

The elements $a_{ij}b_{ij}^{(1)}$ of the $(4-1)$ determinant is as follows:

$$\begin{aligned}
c_{11} &= b_{11}^{(1)} = b_{11} - b_{14}b_{41} = -11 - (0 \times -4) = -11 \\
c_{12} &= b_{12}^{(1)} = b_{12} - b_{14}b_{42} = 4 - (0 \times 3) = 4 \\
c_{13} &= b_{13}^{(1)} = b_{13} - b_{14}b_{43} = -2 - (0 \times 1) = -2
\end{aligned}$$

$$\begin{aligned}
c_{21} &= b_{21}^{(1)} = b_{21} - b_{24}b_{41} = -11 - (-3 \times -4) = -23 \\
c_{22} &= b_{22}^{(1)} = b_{22} - b_{24}b_{42} = 6 - (-3 \times 3) = 15 \\
c_{23} &= b_{23}^{(1)} = b_{23} - b_{24}b_{43} = -5 - (-3 \times 1) = -2
\end{aligned}$$

$$\begin{aligned}
c_{31} &= b_{31}^{(1)} = b_{31} - b_{34}b_{41} = 11 - (1 \times -4) = 15 \\
c_{32} &= b_{32}^{(1)} = b_{32} - b_{34}b_{42} = -5 - (1 \times 3) = -8 \\
c_{33} &= b_{33}^{(1)} = b_{33} - b_{34}b_{43} = 9 - (1 \times 1) = 8
\end{aligned}$$

$$\bar{\triangle}_4 = -(-1)^8 \begin{vmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{vmatrix} = -(-1)^8 \begin{vmatrix} -11 & 4 & -2 \\ -23 & 15 & -2 \\ 15 & -8 & 8 \end{vmatrix}$$

To obtain a pivot element, we shall divide column 3 by -2 to get;

$$\bar{\triangle}_4 = -2(-1)^8 \begin{vmatrix} -11 & 4 & \boxed{1} \\ -23 & 15 & 1 \\ 15 & -8 & -4 \end{vmatrix}$$

Now taking $c_{13} = 1$ for the pivot element, elements $d_{ij} = c_{ij}^{(1)}$ of the $(3-1)$ determinant is as follows:

$$\begin{aligned}
d_{11} &= c_{21}^{(1)} = c_{21} - c_{23}c_{11} = -23 - (1 \times -11) = -12 \\
d_{12} &= c_{22}^{(1)} = c_{22} - c_{23}c_{12} = 15 - (1 \times 4) = 11
\end{aligned}$$

$$\begin{aligned}
d_{21} &= c_{31}^{(1)} = c_{31} - c_{33}c_{11} = 15 - (-4 \times -11) = -29 \\
d_{22} &= c_{32}^{(1)} = c_{32} - c_{33}c_{12} = -8 - (-4 \times 4) = 8
\end{aligned}$$

i.e.,

$$\begin{aligned}
\bar{\triangle}_3 &= -2\begin{vmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{vmatrix} \\
&= -2\begin{vmatrix} -12 & 11 \\ -29 & 8 \end{vmatrix} \\
&= -2[(-12 \times 8) - (11 \times -29)] \\
\therefore \quad \det(A) &= 446
\end{aligned}$$

10.9.3 Properties of Determinants

(i) Interchange of Two Rows or Columns: If two rows or columns of a matrix are interchanged, the sign of the determinant $\triangle_n$ changes accordingly.

for example, $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = -2, \quad \text{and} \quad \begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix} = 2$

Observe that as row 1 and row 2 is interchanged, the determinants changes to 2

(ii) Addition to One Row (Column) of a Multiple of Another Row (Column): The addition of a constant c times row (column) k to row i (or column j) leaves the determinant unchanged.

For example,

$$\begin{vmatrix} 1 & 3 & 5 \\ -2 & -4 & -2 \\ 3 & 5 & -3 \end{vmatrix} \quad \begin{matrix} R_2 \to 2R_1 + R_2 \\ R_3 \to -3R_1 + R_3 \end{matrix}$$

$$= \begin{vmatrix} 1 & 3 & 5 \\ 0 & 2 & 8 \\ 0 & -4 & -18 \end{vmatrix} = 1\begin{vmatrix} 2 & 8 \\ -4 & -18 \end{vmatrix} = -4$$

This property is very important as we can compute the determinant of a matrix by expanding by the column whose non-zero element is one after the elementary transformation.

(iii) Multiplication of the Elements of One Row (Column) by a Non-zero constant: If each of the element of some row or column of a determinants is multiplied by a non-zero factor, α, then the determinant is also multiplied by the same factor, α.

for example,

$$\begin{vmatrix} -2 & 0 & 1 \\ 10 & 3 & -2 \\ 4 & 5 & 3 \end{vmatrix} = 2\begin{vmatrix} -1 & 0 & 1 \\ 5 & 3 & -2 \\ 2 & 5 & 3 \end{vmatrix}$$

(iv) If a Matrix A has two Rows or Column alike: The determinant of a matrix with two rows or columns identical is zero.
for example,

$$|A| = \begin{vmatrix} 3 & 4 & 1 \\ 1 & -5 & 3 \\ 6 & 8 & 2 \end{vmatrix} = \begin{vmatrix} 3 & 4 & 1 \\ 1 & -5 & 3 \\ 0 & 0 & 0 \end{vmatrix} = 0$$

In other words, if two rows or columns are similar, adding a multiple of one to the other reduces the row (column) to all zero and if such exists, the determinant becomes zero. It is obvious as shown above where row three is twice row one (i.e. $R_3 = 2R_1$).

(v) The Determinant of the Product Matrix AB: The determinant of a product of two matrices A and B is equal to product of their respective determinants.
i.e.,

$$\left| \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix} \right| = \begin{vmatrix} 5 & -3 \\ 8 & -4 \end{vmatrix} = 4$$

Conversely,

$$\begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix} \begin{vmatrix} 2 & 0 \\ 1 & -1 \end{vmatrix} = 4$$

$$\therefore \quad |AB| = |A|.|B|$$

(vi) Determinant of a Triangular Matrix: The determinant of a triangular matrix is equal to the product of its principal diagonal elements, namely: if $T = [t_{ij}]$ is a triangular matrix, then we obviously have;

$$\det(T) = t_{11}t_{22}...t_{nn}$$

This property can be used to find the determinant of any given matrix A, by first *decomposing* the matrix into an upper and lower triangular matrices, and hence finding the determinant using the principal diagonal elements as stated above.
For example,

$$\begin{vmatrix} 2 & -3 & 4 \\ 1 & 2 & -3 \\ -1 & -2 & 5 \end{vmatrix} = \begin{vmatrix} 2 & 0 & 0 \\ 1 & \frac{7}{2} & 0 \\ -1 & -\frac{7}{2} & 2 \end{vmatrix} \begin{vmatrix} 1 & -\frac{3}{2} & 2 \\ 0 & 1 & -\frac{10}{7} \\ 0 & 0 & 1 \end{vmatrix}$$
$$= 2 \times \tfrac{7}{2} \times 2$$
$$= 14$$

i.e., if a matrix $A = [a_{ij}]$ is decomposed into two triangular matrices T_1 and T_2, then;

$$|A| = |T_1||T_2| = |T_1|$$

where T_1 is the triangular matrix containing the principal diagonal elements.

(vii) Determinant of a Diagonal Matrix: The determinant of a diagonal matrix $\triangle[d_{ii}]$ is the product of the diagonal elements i.e.,

$$|\triangle| = d_{11}d_{22}...d_{nn}$$

This rule or property also provides a system for computing the determinant $|A|$ of a given matrix $A = [a_{ij}]$. This is performed by reducing A to a diagonal form D, using elementary matrix transformation; by recording the number of interchanges of rows used and the various scalars used to multiply rows (or

columns) of A.
For example,

$$\begin{vmatrix} 5 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 7 \end{vmatrix} = 5 \times (-2) \times 7 = -70$$

(viii) The Determinant of the Transpose of a Matrix: If $A^\top$ is the transpose of a matrix A, then the determinant of the transpose $|A^\top|$ is equal to the determinant of the original matrix A i.e.,

$$|A^\top| = |A|$$

For example,

$$\left|\begin{bmatrix} 1 & 3 & 5 \\ -2 & -4 & -2 \\ 3 & 5 & -3 \end{bmatrix}^\top\right| = \begin{vmatrix} 1 & -2 & 3 \\ 3 & -4 & 5 \\ 5 & -2 & -3 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 3 & 5 \\ -2 & -4 & -2 \\ 3 & 5 & -3 \end{vmatrix} = -4$$

(ix) If matrix A is a square of order n, then
$\det(\alpha A) = \alpha^n \det(A)$, **where α is a constant.**
e.g., let

$$A = \begin{bmatrix} 1 & 3 & 5 \\ -2 & -4 & -2 \\ 3 & 5 & -3 \end{bmatrix} \text{ and } \alpha = 2$$

Then,

$$\begin{aligned} det(\alpha A) &= \left| 2\begin{bmatrix} 1 & 3 & 5 \\ -2 & -4 & -2 \\ 3 & 5 & -3 \end{bmatrix} \right| \\ &= \begin{vmatrix} 2 & 6 & 10 \\ -4 & -8 & -4 \\ 6 & 10 & -6 \end{vmatrix} \\ &= \begin{vmatrix} 2 & 6 & 10 \\ 0 & 4 & 16 \\ 0 & 0 & -4 \end{vmatrix} \\ &= 2 \times 4 \times -4 \\ &= -32 \end{aligned}$$

Alternatively, using the property; $det(\alpha A) = \alpha^n det(A)$, we have;

$$det(\alpha A) = \alpha^3 det(A) = 2^3 \begin{vmatrix} 1 & 3 & 5 \\ -2 & -4 & -2 \\ 3 & 5 & -3 \end{vmatrix} = 2^3 \times -4 = -32$$

$$\therefore \qquad det(\alpha A) = \alpha^n det(A) = 2^3 det(A) = -32$$

10.9.4 Factorization of Determinants

Example 10.9.5

Solve for z in the following equation: $\begin{vmatrix} z-1 & 4 & -1 \\ 1 & z+2 & 1 \\ 2z-4 & 4 & z-4 \end{vmatrix} = 0$

Solution

$$\begin{vmatrix} z-1 & 4 & -1 \\ 1 & z+2 & 1 \\ 2z-4 & 4 & z-4 \end{vmatrix} C_1 \to -C_3 + C_1$$

$$= \begin{vmatrix} z & 4 & -1 \\ 0 & z+2 & 1 \\ z & 4 & z-4 \end{vmatrix} = z \begin{vmatrix} 1 & 4 & -1 \\ 0 & z+2 & 1 \\ 1 & 4 & z-4 \end{vmatrix} R_3 \to -R_1 + R_3$$

$$= z\begin{vmatrix} \boxed{1} & 4 & -1 \\ 0 & z+2 & 1 \\ 0 & 0 & z-3 \end{vmatrix}$$

Expanding through the boxed element, (1), we have;

$$z\begin{vmatrix} z+2 & 1 \\ 0 & z-3 \end{vmatrix} = z(z+2)(z-3) = 0$$

Thus, solving, we have; $z = 0, -2$ or 3

Example 10.9.6

Factories the determinant: $\begin{vmatrix} x & y & z \\ y+z & z+x & x+y \\ yz & zx & xy \end{vmatrix}$

Solution

$$\begin{vmatrix} x & y & z \\ y+z & z+x & x+y \\ yz & zx & xy \end{vmatrix} C_1 \to -C_3 + C_1$$

$$= \begin{vmatrix} x-z & y & z \\ z-x & z+x & x+y \\ y(z-x) & zx & xy \end{vmatrix}$$

$$= (z-x)\begin{vmatrix} -1 & y & z \\ 1 & z+x & x+y \\ y & zx & xy \end{vmatrix} \begin{matrix} R_1 \to \frac{R_3}{y} + R_1 \\ R_2 \to -\frac{R_3}{y} + R_2 \end{matrix}$$

$$= (z-x)\begin{vmatrix} 0 & \frac{zx}{y} + y & z+x \\ 0 & -\frac{zx}{y} + (z+x) & y \\ \boxed{\text{y}} & zx & xy \end{vmatrix}$$

Now expanding through the boxed element, y, we have;

$$y(z-x)\begin{vmatrix} \frac{zx+y^2}{y} & z+x \\ \frac{yx+zy-zx}{y} & y \end{vmatrix} = (z-x)\begin{vmatrix} zx+y^2 & z+x \\ yx+zy-zx & y \end{vmatrix}$$

$$\begin{aligned}
&= (z-x)[y(zx+y^2)-(z+x)(yx+zy-zx)] \\
&= (z-x)[yzx+y^3-(zyx+z^2y-z^2x+x^2y+xzy-zx^2) \\
&= (z-x)[yzx+y^3-zyx-z^2y+z^2x-x^2y-xzy+zx^2] \\
&= (z-x)[x^2(z-y)+y(y^2-z^2)+zx(z-y)] \\
&= (z-x)(z-y)[x^2-y(z+y)+zx] \\
&= (z-x)(z-y)[x^2-yz-y^2+zx] \\
&= (z-x)(z-y)[x^2+xz-yz-y^2] \\
&= (z-x)(z-y)(y-x)(x+y+z)
\end{aligned}$$

$$\therefore \quad \begin{vmatrix} x & y & z \\ y+z & z+x & x+y \\ yz & zx & xy \end{vmatrix} = (z-x)(z-y)(y-x)(x+y+z)$$

10.10 Inverse of a Non-Singular Matrix

The inverse of a given matrix is a matrix such that, when multiplied on the right (post-multiplied) or on the left (premultiplied) by the given matrix, yields the unit matrix I.

We denote the inverse of a given matrix A by A^{-1}. Thus, by definition, we have;

$$AA^{-1} = A^{-1}A = I \tag{10.10.1}$$

where I is the unit or identity matrix. Finding the inverse of a given matrix is called *matrix inversion*.

By non-singular matrix, we mean a matrix whose determinant is not equal to zero. In other words, a square matrix is termed *non-singular* or *invertible* if the determinant is not equal to zero, otherwise it is called a *singular matrix* or *non-invertible*

Theorem 10.10.1. *Every non-singular matrix has an inverse.*

Proof. Suppose we have a non-singular matrix, $A = [a_{ij}]_{n,n}$, such that;

$$A = \begin{bmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ ... & ... & ... & ... \\ a_{n1} & a_{n2} & ... & a_{nn} \end{bmatrix}$$

where $\det(A) = \triangle \neq 0$. We form the *adjoint of matrix A* as follows:

$$adj(A) = \begin{bmatrix} A_{11} & A_{21} & ... & A_{n1} \\ A_{12} & A_{22} & ... & A_{n2} \\ ... & ... & ... & ... \\ A_{1n} & A_{2n} & ... & A_{nn} \end{bmatrix} \tag{10.10.2}$$

where A_{ij} are the cofactors (i.e. signed minors) of the corresponding elements a_{ij}; $(i, j = 1, 2, ..., n)$. The adjoint of a matrix will be better appreciated with examples in the next section when we shall be discussing the inverse of a matrix.

However from equation 10.10.2, observe that the cofactors of the elements of the rows lie in the corresponding columns, that is, the transposed operation is performed.

Now, dividing all the elements of the adjoint matrix by the value of the determinant of A, we have;

$$A^* = \begin{bmatrix} \frac{A_{11}}{\triangle} & \frac{A_{21}}{\triangle} & ... & \frac{A_{n1}}{\triangle} \\ \frac{A_{12}}{\triangle} & \frac{A_{22}}{\triangle} & ... & \frac{A_{n2}}{\triangle} \\ ... & ... & ... & ... \\ \frac{A_{1n}}{\triangle} & \frac{A_{2n}}{\triangle} & ... & \frac{A_{nn}}{\triangle} \end{bmatrix} \tag{10.10.3}$$

We shall prove that the matrix A^* is the required inverse such that,

$$A^* = A^{-1}$$

Using the properties of 10.10.1 and 10.10.2, we multiply AA^* to get:

$$\begin{aligned} AA^* &= \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} \frac{A_11}{\triangle} & \frac{A_{21}}{\triangle} & \dots & \frac{A_{n1}}{\triangle} \\ \frac{A_{21}}{\triangle} & \frac{A_{22}}{\triangle} & \dots & \frac{A_{n2}}{\triangle} \\ \dots & \dots & \dots & \dots \\ \frac{A_{n1}}{\triangle} & \frac{A_{n2}}{\triangle} & \dots & \frac{A_{nn}}{\triangle} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \end{bmatrix} = I \end{aligned} \tag{10.10.5}$$

The above formula can be derived faster if we use the compact notation as follows:

$$A = [a_{ij}] \text{ and } A^* = \left[\frac{A_{ji}}{\triangle}\right]$$

Taking expression (10.10.4) into account, we obtain;

$$AA^* = \left[\sum_{k=1}^{n} a_{ik}\frac{A_{jk}}{\triangle}\right] = [\delta_{ij}] = I$$

Similarly, we observe that $A^*A = I$
Hence,

$$A^* = A^{-1}$$

or

$$A^{-1} = \frac{1}{\triangle}[A_{ji}] \tag{10.10.6}$$

where $\triangle = \det(A)$ and A_{ji}= adjoint (A). □

Corollary 10.10.2. *The inverse A^{-1} of a given matrix A is unique. Thus, every right inverse (left inverse) of A coincides with its inverse A^{-1} (if such exists). However, unlike in elementary arithmetic, $A^{-1} \neq \frac{1}{A}$ in matrices; you must have to follow the computations of eqn. 10.10.6 to arrive at the desired solution.*

Thus, if

$$AB = I$$

then, premultiplying this equation by A^{-1} *we obtain;*

$$A^{-1}AB = A^{-1}I$$

or

$$B = A^{-1}$$

We similarly prove that if,

$$CA = I$$

then,

$$C = A^{-1}$$

Therefore, when verifying expression (10.10.1) one equation is sufficient.

Corollary 10.10.3. *A square matrix* A *is non-singular if and only if its rows are linearly independent.*

Proof. Each matrix A determines a corresponding linear transformation $Y = XA$ which is non-singular if and only if the matrix A is non-singular. The result then follows from the theorem. □

Corollary 10.10.4. *A singular matrix does not have an inverse.*

Proof. If a square matrix A is singular, then it has no left-inverse and no right-inverse, and the rows and also the columns of A are linearly dependent. Since the matrix A is singular, then,

$$\det(A) = 0$$

From (10.10.1), we have;

$$\det(A). \det(A^{-1}) = \det(I) = 1$$

or

$$0 = 1(?!)$$

which is impossible; which completes the proof. □

10.10.1 Inverse of a 2×2 Matrix

If A is any 2×2 matrix of the form: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then A is invertible if there exists a $B = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$ such that $AB = I = BA$.

Observe that if $AB = I$, then;

$$\begin{bmatrix} x & y \\ z & w \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\text{i.e.,} \quad \begin{bmatrix} xa + yc & xb + yd \\ za + wc & zb + wd \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Equating corresponding entries, we obtain;

$$\begin{array}{lcl} xa + yc = 1, & & xb + yd = 0 \\ za + wc = 0 & \text{and} & zb + wd = 1 \end{array}$$

Solving these equations in pairs for x, y, z and w, we have;

$$x = \frac{d}{ad - bc}, \quad y = \frac{-b}{ad - bc}$$

$$z = \frac{-c}{ad - bc}, \quad w = \frac{a}{ad - bc}$$

where $ad - bc = |A|$
So that the inverse of the matrix A is:

$$B = \begin{bmatrix} \frac{d}{|A|} & \frac{-b}{|A|} \\ \frac{-c}{|A|} & \frac{a}{|A|} \end{bmatrix}$$

which we can write as;

$$A^{-1} = \frac{1}{|A|} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

where $\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ is the adjoint of the matrix A.

Example 10.10.1

Compute the inverse of the 2×2 matrix, (i) $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$,
(ii) $\begin{bmatrix} -1 & -3 \\ 5 & 7 \end{bmatrix}$

Solution

(i) If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, then the $\det(A)$ can be obtained by;

$$|A| = 4 - 6 = -2 \neq 0$$

$\therefore$ A is non-singular.
Hence the Adjoint of A;

$$adj(A) = \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$$

$$\therefore \quad A^{-1} = \frac{1}{|A|} adj(A) = \frac{1}{-2} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$$

(ii) $A = \begin{bmatrix} -1 & -3 \\ 5 & 7 \end{bmatrix}$
Thus,

$|A| = -7 - (-15) = -7 + 15 = 8 \neq 0$ (i.e. A is non-singular)

Thus,

$$adj(A) = \begin{bmatrix} 7 & 3 \\ -5 & -1 \end{bmatrix}$$

$$\therefore \quad A^{-1} = \frac{1}{|A|} adj(A) = \frac{1}{8} \begin{bmatrix} 7 & 3 \\ -5 & -1 \end{bmatrix}$$

10.10.2 Inverse of Higher Order, $n > 2$

Example 10.10.2

Find the inverse of the matrix: $A = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$

Solution

$A^{-1} = \frac{1}{|A|}$ adj (A) provided that $|A| \neq 0$ (i.e. A is non-singular).

Observe that adjoint of A is the transpose of the cofactors of A. Now, computing for the cofactors, we have;

$$A_{ij} = (-1)^{i+j}|M_{ij}|$$

where M_{ij} are the minors of A. Thus,

$$A_{11} = (-1)^{1+1}\begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix} = 1; \quad A_{12} = (-1)^{1+2}\begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix} = -1$$

$$A_{13} = (-1)^{4}\begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} = 0; \quad A_{21} = (-1)^{3}\begin{vmatrix} 2 & 6 \\ 2 & 5 \end{vmatrix} = 2$$

$$A_{22} = (-1)^{4}\begin{vmatrix} 3 & 6 \\ 2 & 5 \end{vmatrix} = 3; \quad A_{23} = (-1)^{5}\begin{vmatrix} 3 & 2 \\ 2 & 2 \end{vmatrix} = -2$$

$$A_{31} = (-1)^{4}\begin{vmatrix} 2 & 6 \\ 1 & 2 \end{vmatrix} = -2; \quad A_{32} = (-1)^{5}\begin{vmatrix} 3 & 6 \\ 1 & 2 \end{vmatrix} = 0$$

$$A_{33} = (-1)^{6}\begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix} = 1$$

Thus, the matrix of cofactors is obtained as;

$$[A_{ij}] = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & -2 \\ -2 & 0 & 1 \end{bmatrix}$$

and

$$\begin{aligned} adj(A) = [A_{ji}] &= [A_{ij}]^{\top} \\ &= \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} \end{aligned}$$

But

$$\begin{aligned} |A| &= \textstyle\sum_{j=1}^{3} a_{ij}A_{ij} \\ &= a_{11}A_{11} + a_{12}A_{12} + A_{13}A_{13} \\ &= 3(1) + 2(-1) + 6(0) = 1 \end{aligned}$$

$$\therefore \quad A^{-1} = \frac{1}{|A|} adj(A)$$

$$= \frac{1}{1}\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

You are advised to verify that we indeed have $A^{-1}A = AA^{-1} = I$.

10.10.3 Inverse of a Non-Singular Matrix Using Elementary Row Operations

A matrix obtained from the identity matrix by means of elementary row operations is called an *elementary matrix* (Section 10.5). Thus, another method of finding inverses of non-singular matrices is by elementary operations or linear transformation on the given matrix as well as on the identity matrix of the same order simultaneously until the given matrix becomes the identity matrix.

Theorem 10.10.5. *Suppose A is invertible and say it is row reducible to the identity matrix by sequence of elementary row operations, then this sequence of elementary row operations applied to I yields A^{-1}.*

The proof of this theorem shall be illustrated by example, however, you are advised to refer back to section 10.4 for elementary row, operations of matrix incase of any difficulty.

Example 10.10.3

Use elementary row operations to obtain the inverse of the non-singular matrix:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -2 & -4 & -5 \\ 3 & 5 & 6 \end{bmatrix}$$

Solution

Form the matrix $[A : I]$ row reduce by elementary row operations to row canonical form;

$$[A : I] = \left[\begin{array}{rrrcrrr} 1 & 2 & 3 & : & 1 & 0 & 0 \\ -2 & -4 & -5 & : & 0 & 1 & 0 \\ 3 & 5 & 6 & : & 0 & 0 & 1 \end{array}\right] \begin{array}{l} R_2 \rightarrow 2R_1 + R_2 \\ R_3 \rightarrow -3R_1 + R_3 \end{array}$$

$$= \left[\begin{array}{rrrcrrr} 1 & 2 & 3 & : & 1 & 0 & 0 \\ 0 & 0 & 1 & : & 2 & 1 & 0 \\ 0 & -1 & -3 & : & -3 & 0 & 1 \end{array}\right] R_2 \longleftrightarrow R_3$$

$$= \left[\begin{array}{rrrcrrr} 1 & 2 & 3 & : & 1 & 0 & 0 \\ 0 & -1 & -3 & : & -3 & 0 & 1 \\ 0 & 0 & 1 & : & 2 & 1 & 0 \end{array}\right] R_2 \rightarrow -R_2$$

$$= \left[\begin{array}{rrrcrrr} 1 & 2 & 3 & : & 1 & 0 & 0 \\ 0 & 1 & 3 & : & 3 & 0 & -1 \\ 0 & 0 & 1 & : & 2 & 1 & 0 \end{array}\right] \begin{array}{l} R_1 \rightarrow -2R_2 + R_1 \\ R_2 \rightarrow -3R_3 + R_2 \end{array}$$

$$= \left[\begin{array}{rrrcrrr} 1 & 0 & -3 & : & -5 & 0 & 2 \\ 0 & 1 & 0 & : & -3 & -3 & -1 \\ 0 & 0 & 1 & : & 2 & 1 & 0 \end{array}\right] R_1 \rightarrow 3R_3 + R_1$$

$$= \left[\begin{array}{rrrcrrr} 1 & 0 & 0 & : & 1 & 3 & 2 \\ 0 & 1 & 0 & : & -3 & -3 & -1 \\ 0 & 0 & 1 & : & 2 & 1 & 0 \end{array}\right]$$

Note that the final block matrix is of the form $[I : B]$, where I is the identity matrix. Hence A is invertible and B is its inverse; thus,

$$A^{-1} = \left[\begin{array}{rrr} 1 & 3 & 2 \\ -3 & -3 & -1 \\ 2 & 1 & 0 \end{array}\right]$$

Example 10.10.4

Find the inverse of $A = \begin{bmatrix} 3 & 4 & 5 \\ 1 & -1 & 2 \\ 2 & 1 & 3 \end{bmatrix}$ using elementary row operations and hence show that the inverse of A, is a product of the elementary matrices $E_nE_{n-1}...E_1$, corresponding to the e.r.o.s., $\varepsilon_n\varepsilon_{n-1}...\varepsilon_1$ such that,

$$A^{-1} = E_nE_{n-1}...E_1$$

Solution

$$\begin{aligned}
\text{A} : I &= \begin{bmatrix} 3 & 4 & 5 & : & 1 & 0 & 0 \\ 1 & -1 & 2 & : & 0 & 1 & 0 \\ 2 & 1 & 3 & : & 0 & 0 & 1 \end{bmatrix} R_1 \longleftrightarrow R_2 : \varepsilon_1 \\
&= \begin{bmatrix} 1 & -1 & 2 & : & 0 & 1 & 0 \\ 3 & 4 & 5 & : & 1 & 0 & 0 \\ 2 & 1 & 3 & : & 0 & 0 & 1 \end{bmatrix} \begin{matrix} R_2 \to -3R_1 + R_2 : \varepsilon_2 \\ R_3 \to -2R_1 + R_3 : \varepsilon_3 \end{matrix} \\
&= \begin{bmatrix} 1 & -1 & 2 & : & 0 & 1 & 0 \\ 0 & 7 & -1 & : & 1 & -3 & 0 \\ 0 & 3 & -1 & : & 0 & -2 & 1 \end{bmatrix} R_3 \to \tfrac{-3}{7}R_2 + R_3 : \varepsilon_4 \\
&= \begin{bmatrix} 1 & -1 & 2 & : & 0 & 1 & 0 \\ 0 & 7 & -1 & : & 1 & -3 & 0 \\ 0 & 0 & -\frac{4}{7} & : & -\frac{3}{7} & -\frac{5}{7} & 1 \end{bmatrix} \begin{matrix} R_1 \to \frac{1}{7}R_2 + R_1 : \varepsilon_5 \\ R_2 \to \frac{1}{7}R_2 : \varepsilon_6 \end{matrix} \\
&= \begin{bmatrix} 1 & 0 & \frac{13}{7} & : & \frac{1}{7} & \frac{4}{7} & 0 \\ 0 & 1 & -\frac{1}{7} & : & \frac{1}{7} & -\frac{3}{7} & 0 \\ 0 & 0 & -\frac{4}{7} & : & -\frac{3}{7} & -\frac{5}{7} & 1 \end{bmatrix} \begin{matrix} R_1 \to \frac{13}{4}R_3 + R_1 : \varepsilon_7 \\ R_2 \to -\frac{1}{4}R_3 + R_2 : \varepsilon_8 \\ R_3 \to -\frac{7}{4}R_3 : \varepsilon_9 \end{matrix} \\
&= \begin{bmatrix} 1 & 0 & 0 & : & -\frac{5}{4} & -\frac{7}{4} & \frac{13}{4} \\ 0 & 1 & 0 & : & \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ 0 & 0 & 1 & : & \frac{3}{4} & \frac{5}{4} & -\frac{7}{4} \end{bmatrix}
\end{aligned}$$

$$\therefore \qquad A^{-1} = \frac{1}{4}\begin{bmatrix} -5 & -7 & 13 \\ 1 & -1 & -1 \\ 3 & 5 & -7 \end{bmatrix}$$

Therefore, the elementary matrices $E_i, i = 1(n)$ corresponding to these e.r.o.s $\varepsilon_i, i = 1(n)$ respectively are obtained by performing each ε on I_3, where,

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

as follows:

$$E_1 : R_1 \longleftrightarrow R_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_2 : R_2 \to -3R_1 + R_2 = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_3 : R_3 \to -2R_1 + R_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

$$E_4 : R_3 \to -\frac{3}{7}R_2 + R_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{3}{7} & 1 \end{bmatrix}$$

$$E_5 : R_1 \to \frac{1}{7}R_2 + R_1 = \begin{bmatrix} 1 & \frac{1}{7} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_6 : R_2 \to \frac{1}{7}R_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{7} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_7 : R_1 \to \frac{13}{4}R_3 + R_1 = \begin{bmatrix} 1 & 0 & \frac{13}{4} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_8 : R_2 \to -\frac{1}{4}R_3 + R_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{4} \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_9 : R_3 \to -\frac{7}{4}R_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{7}{4} \end{bmatrix}$$

Thus, to show that A^{-1} is a product of the elementary matrices $E_n E_{n-1} \ldots E_1$, we have;

$$A^{-1} = E_9 E_8 E_7 E_6 E_5 E_4 E_3 E_2 E_1$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{7}{4} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{4} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{4} \\ 0 & 0 & -\frac{7}{4} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{4} \\ 0 & 0 & -\frac{7}{4} \end{bmatrix} \begin{bmatrix} 1 & 0 & \frac{13}{4} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & \frac{13}{4} \\ 0 & 1 & -\frac{1}{4} \\ 0 & 0 & -\frac{7}{4} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & \frac{13}{4} \\ 0 & 1 & -\frac{1}{4} \\ 0 & 0 & -\frac{7}{4} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{7} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & \frac{13}{4} \\ 0 & \frac{1}{7} & -\frac{1}{4} \\ 0 & 0 & -\frac{7}{4} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & \frac{13}{4} \\ 0 & \frac{1}{7} & -\frac{1}{4} \\ 0 & 0 & -\frac{7}{4} \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{7} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{7} & \frac{13}{4} \\ 0 & \frac{1}{7} & -\frac{1}{4} \\ 0 & 0 & -\frac{7}{4} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \frac{1}{7} & \frac{13}{4} \\ 0 & \frac{1}{7} & -\frac{1}{4} \\ 0 & 0 & -\frac{7}{4} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{3}{7} & 1 \end{bmatrix} = \begin{bmatrix} 1 & -\frac{5}{4} & \frac{13}{4} \\ 0 & \frac{1}{4} & -\frac{1}{4} \\ 0 & \frac{3}{4} & \frac{-7}{4} \end{bmatrix}$$

$$\text{i.e.,}\quad A^{-1} = \begin{bmatrix} 1 & -\frac{5}{4} & \frac{13}{4} \\ 0 & \frac{1}{4} & -\frac{1}{4} \\ 0 & \frac{3}{4} & -\frac{7}{4} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{11}{2} & -\frac{5}{4} & \frac{13}{4} \\ \frac{1}{2} & \frac{1}{4} & -\frac{1}{4} \\ \frac{7}{2} & \frac{3}{4} & -\frac{7}{4} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{11}{2} & -\frac{5}{4} & \frac{13}{4} \\ \frac{1}{2} & \frac{1}{4} & -\frac{1}{4} \\ \frac{7}{2} & \frac{3}{4} & -\frac{7}{4} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{-7}{4} & -\frac{-5}{4} & \frac{13}{4} \\ -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \\ \frac{5}{4} & \frac{3}{4} & -\frac{7}{4} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{7}{4} & -\frac{5}{4} & \frac{13}{4} \\ -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \\ \frac{5}{4} & \frac{3}{4} & -\frac{7}{4} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{5}{4} & -\frac{7}{4} & \frac{13}{4} \\ \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ \frac{3}{4} & \frac{5}{4} & -\frac{7}{4} \end{bmatrix}$$

$$A^{-1} = \frac{1}{4} \begin{bmatrix} -5 & -7 & 13 \\ 1 & -1 & -1 \\ 3 & 5 & -7 \end{bmatrix}$$

Corollary 10.10.6. *Let A be a non-singular matrix and $\varepsilon_n \varepsilon_{n-1} ... \varepsilon_1 A = I_n$ (10.4.3) be elementary row operations (e.r.o.s) on A,, then the inverse of the matrix A is the product of the elementary matrices such that,*

$$A^{-1} = E_n E_{n-1} ... E_1 \quad \textit{(note the order!)} \tag{10.10.8}$$

where $E_i, i = 1(n)$ is the elementary matrices corresponding to the e.r.o.s $\varepsilon_i, i = 1(n)$ on A.

10.10.4 Basic Properties of Inverse Matrix

(i) The determinant of an inverse matrix is equal to the reciprocal of the determinant of the original matrix. Suppose,

$$A^{-1} A = I$$

Taking into consideration that the determinant of a product of two square matrices is equal to the product of the determinants of the matrices, we have;

$$\det(A^{-1}).\det(A) = \det(I) = 1$$

Hence,

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

Example 10.10.5

Let $A = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}$

Then,

$$\det(A) = 6 - 4 = 2$$

and

$$A^{-1} = \frac{1}{|A|} adj(A) = \frac{1}{2} \begin{bmatrix} 3 & -1 \\ -4 & 2 \end{bmatrix}$$

Thus,

$$\det(A^{-1}) = \left(\frac{1}{2}\right)^2 \begin{vmatrix} 3 & -1 \\ -4 & 2 \end{vmatrix} = \frac{1}{4}(6-4) = \frac{1}{2}$$

$$\therefore \quad \det(A^{-1}) = \frac{1}{\det(A)}$$

(ii) The inverse of a product of square matrices is equal to the product of the inverses of the factors taken in reverse order:

$$(AB)^{-1} = B^{-1}A^{-1} \text{ (note the order!)}$$

Indeed,

$$AB(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$$

and

$$(B^{-1}A^{-1})AB = B^{-1}(A^{-1}A)B = B^{-1}IB = B^{-1}B = I$$

Hence, $B^{-1}A^{-1}$ is the inverse of AB.

However, in a more general case,

$$(A_1A_2...A_p)^{-1} = A_p^{-1}A_{p-1}^{-1}...A_1^{-1}$$

Example 10.10.6

Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$; $\quad B = \begin{bmatrix} -3 & 1 \\ 5 & 6 \end{bmatrix}$
Then,

$$(AB)^{-1} = \begin{bmatrix} 7 & 13 \\ 11 & 27 \end{bmatrix}^{-1} = \frac{1}{46}\begin{bmatrix} 27 & -13 \\ -11 & 7 \end{bmatrix}$$

and

$$\begin{aligned} B^{-1}A^{-1} &= -\tfrac{1}{23}\begin{bmatrix} 6 & -1 \\ -5 & -3 \end{bmatrix}\left(-\tfrac{1}{2}\right)\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} \\ &= \tfrac{1}{46}\begin{bmatrix} 27 & -13 \\ -11 & 7 \end{bmatrix} \\ &= (AB)^{-1} \end{aligned}$$

(iii) The transpose of an inverse is equal to the inverse of the transpose of the given matrix:

$$(A^{-1})^{\top} = (A^{\top})^{-1}$$

Taking the transpose of $A^{-1}A = I$, we obtain;

$$(A^{-1}A)^{\top} = A^{\top}(A^{-1})^{\top} = I^{\top} = I$$

Hence, pre-multiplying the last equation by the matrix, $(A^{\top})^{-1}$, we get;

$$(A^{\top})^{-1}A^{\top}(A^{-1})^{\top} = (A^{\top})^{-1}I$$

or

$$(A^{-1})^{\top} = (A^{\top})^{-1}$$

which is what we set to prove. Therefore $A^{\top}$ will have an inverse whenever A does, so that what was proved about the independence of rows of A earlier on also applies equally to its columns.

Example 10.10.7

Let $A = \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix}$
then,

$$(A^{-1})^\top = \left\{ \frac{1}{10} \begin{bmatrix} 4 & 2 \\ -3 & 1 \end{bmatrix} \right\}^\top = \frac{1}{10} \begin{bmatrix} 4 & -3 \\ 2 & 1 \end{bmatrix}$$

and

$$(A^\top)^{-1} = \begin{bmatrix} 1 & 3 \\ -2 & 4 \end{bmatrix}^{-1} = \frac{1}{10} \begin{bmatrix} 4 & -3 \\ 2 & 1 \end{bmatrix}$$

$$\therefore \qquad (A^{-1})^\top = (A^\top)^{-1}$$

Remark 27 *The matrix equations:*

$$AX = B \text{ and } YA = B$$

are easily solved by means of an inverse matrix. If the $\det(A) \neq 0$*, then;*

$$X = A^{-1}B \text{ and } Y = BA^{-1}$$

10.10.5 Orthogonal Matrices

An n-square matrix A is described as orthogonal if and only if,

$$A^\top A = AA^\top = I_n \tag{10.10.9}$$

(Note that in this case $A^\top$ and A are inverses of each other). Thus another way of putting this is that an orthogonal matrix is an invertible matrix whose inverse is equal to its transpose i.e.,

$$A^{-1} = A^\top \tag{10.10.10}$$

Example 10.10.8

Show that the matrix $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ is orthogonal.

Solution

If A is orthogonal, then;

$$A^\top A = AA^\top \text{ i.e. } A^\top = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

So that,

$$A^\top A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$AA^\top = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\therefore \quad A^\top A = AA^\top$$

Hence A is orthogonal.
Alternatively, using $A^{-1} = A^\top$, we have;

$$A^{-1} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} = A^\top$$

$$A^{-1} = A^\top$$

10.11 Powers of a Matrix

Let A be a square matrix, and p a natural number, then;

$$AA...A = A^p \qquad (10.11.1)$$

Similarly, it also follows from the law of indices, that;

$$A^\circ = I$$

where I is an identity matrix. Furthermore, if matrix A is non-singular, we can introduce a negative power and define the relation by;

$$A^{-p} = (A^{-1})^p \qquad (10.11.2)$$

In other words, the ordinary rules of indices hold for powers of matrices with integral exponents:

(i) $A^pA^q = A^{p+q}$

(ii) $(A^p)^q = A^{pq}$

Example 10.11.1

Let

$$A = \begin{bmatrix} d_1 & 0 & ... & 0 \\ 0 & d_2 & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & d_n \end{bmatrix}$$

be a diagonal matrix of order n, then

$$A^p = \begin{bmatrix} d_1^p & 0 & ... & 0 \\ 0 & d_2^p & ... & 0 \\ ... & .... & ... & ... \\ 0 & 0 & ... & d_n^p \end{bmatrix}$$

Solution

Suppose $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Then,

(i)

$$A^2 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Next,

$$A^3 = A^2 \times A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\therefore \quad A^p A^q = A^{p+q}$
where $p = 2$ and $q = 1$.

(ii) From (i), $A^2 = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $A^3 = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Thus,

$$\begin{aligned}
(A^2)^3 &= \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 4 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 6 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
\therefore \quad A^6 &= \begin{bmatrix} 1 & 6 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\end{aligned}$$

which follows that $(A^2)^3 = A^6$, where $p = 2$ and $q = 3$.

At a close observation of the computation above, we can deduce that;

$$\begin{aligned}
A^p &= A_1 A_2 \ldots A_p \\
&= \begin{bmatrix} 1 & p & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\end{aligned}$$

Remark 28 *If A and B are square matrices of one and the same order, and A and B are commutative over multiplication i.e. $AB = BA$, then the binomial expression holds true; such that,*

$$(A + B)^p = \sum_{k=0}^{p} C_p^k A^k B^{p-k} \tag{10.11.3}$$

We shall illustrate the proof of this relation by example.

Example 10.11.2

Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}$ be matrices then by the assertion, A and B need to be commutative over multiplication such that;

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 0 & 2 \end{bmatrix}$$

Clearly, they are commutative over multiplication and thus $AB = BA$.

In the same vein,

$$\begin{aligned}
(A+B)^2 &= \left\{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}\right\}^2 = \begin{bmatrix} 3 & 4 \\ 0 & 3 \end{bmatrix}^2 \\
&= \begin{bmatrix} 3 & 4 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 3 & 4 \\ 0 & 3 \end{bmatrix} \\
&= \begin{bmatrix} 9 & 24 \\ 0 & 9 \end{bmatrix}
\end{aligned}$$

But,

$$\begin{aligned}
\textstyle\sum_{k=0}^{2} {}^2C_k A^k B^{2-k} &= {}^2C_0 \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}^2 + {}^2C_1 \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix} \\
&\quad + {}^2C_2 \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^2 \\
&= \begin{bmatrix} 4 & 12 \\ 0 & 4 \end{bmatrix} + 2\begin{bmatrix} 2 & 5 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 4+4+1 & 12+10+2 \\ 0 & 4+4+1 \end{bmatrix} \\
&= \begin{bmatrix} 9 & 24 \\ 0 & 9 \end{bmatrix}
\end{aligned}$$

$$\therefore \qquad (A+B)^2 = \sum_{k=0}^{2} {}^2C_k A^k B^{2-k}$$

Provided that $AB = BA$.

10.11.1 Idempotent Matrices

Let A be an n-square matrix, then A is said to be an *idempotent* matrix if,

$$A^2 = A \tag{10.11.4}$$

. Obviously, the identity matrix I_n is a trivial case of an idempotent matrix.

Example 10.11.3

Show that the matrix: $A = \begin{bmatrix} -1 & 2 \\ -1 & 2 \end{bmatrix}$ is an idempotent matrix.

Solution

$$\begin{aligned} A^2 &= \begin{bmatrix} -1 & 2 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ -1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 2 \\ -1 & 2 \end{bmatrix} \\ \therefore \quad A^2 &= A \end{aligned}$$

Hence A is an idempotent matrix.

Remark 29 *For any $n \times n$ matrix,*

$$A_n = \begin{bmatrix} \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \\ \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \end{bmatrix} \textit{ is idempotent.}$$

You may decide to check to see that this is true.

10.11.2 Nilpotent Matrices

A square matrix A is said to be *Nilpotent* if,

$$A^k = 0; \text{ but } A \neq 0 \text{ for } k \in N \tag{10.11.5}$$

where k is called the index of A, and the minimum value of k is the least of all such $k's$.

Example 10.11.4

Show that the matrix: $A = \begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix}$ is Nilpotent.

Solution

$$A^2 = \begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\therefore \quad A$ is Nilpotent since $A^2 = 0; \quad A \neq 0$.
Here, the index of A is $k = 2$.

Note: A 2×2 real nilpotent matrices are of index at most 2.

10.12 Rational Functions of a Matrix

Let $X = [x_{ij}]$ be an arbitrary square matrix of order n, such that;

$$X = \begin{bmatrix} x_{11} & x_{12} & \ldots & x_{1n} \\ x_{21} & x_{22} & \ldots & x_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ x_{1n} & x_{n2} & \ldots & x_{nn} \end{bmatrix}$$

By analogy with the formulas of elementary algebra, we determine the integral rational functions of the matrix X:

$$P(x) = A_0 x^m + A_1 x^{m-1} + \ldots + A_m I \text{ (right polynomial)} \quad (10.12.1)$$

$$\tilde{P}(x) = x^m A_0 + x^{m-1} A_1 + \ldots + I A_m \text{ (left polynomial)} \quad (10.12.2)$$

where $A_k (k = 0, 1, \ldots, m)$ are $m \times n$ or respectively $n \times m$ matrices and I is a unit matrix of order n.

Generally,

$$P(x) \neq \tilde{P}(x)$$

It is also possible to introduce fractional rational functions of the matrix X, defining them by the formulas

$$R_1(x) = P(x)[Q(x)]^{-1} \quad (10.12.3)$$

and

$$R_2(x) = [Q(x)]^{-1} P(x) \quad (10.12.4)$$

where $P(x)$ and $Q(x)$ are matrix polynomials and $\det[Q(x)] \neq 0$.

Example 10.12.1

Let $P(\chi) = \chi^2 + \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \chi - \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ where χ is a variable matrix of order two. Find $p\left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\right)$

Solution

$$\begin{aligned} P\left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\right) &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^2 + \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

Example 10.12.2

Consider the matrix: $\chi = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}$. Show that $\chi^2 - 3\chi I + 3I = 0$; where I and 0 are respectively the 2×2 identity matrix and zero matrices.

Solution

$$
\begin{aligned}
\chi^2 - 3\chi I + 3I &= \chi^2 - 3\chi + 3I \\
&= \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}^2 - 3\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} + 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 3 & 3 \\ -3 & 0 \end{bmatrix} - \begin{bmatrix} 6 & 3 \\ -3 & 3 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \\
&= \begin{bmatrix} 3-6+3 & 3-3+0 \\ -3+3+0 & 0-3+3 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\
&= 0
\end{aligned}
$$

$$\therefore \quad \chi^2 - 3\chi I + 3I = 0$$

10.13 Partitioned Matrices

Let a matrix $A = [a_{ij}]_{m,n}$. It can be *partitioned* or divided into matrices of lower orders (*Sub matrices, or blocks*) by using horizontal or vertical partitions that runs through the whole matrix. For example;

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \vdots & a_{14} \\ a_{21} & a_{22} & a_{23} & \vdots & a_{24} \\ \dots & \dots & \dots & \vdots & \dots \\ a_{31} & a_{32} & a_{33} & \vdots & a_{34} \end{bmatrix}$$

where the partitioned blocks are the sub matrices, where,

$$P = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}, Q = \begin{bmatrix} a_{14} \\ a_{24} \end{bmatrix}, R = [a_{31} \quad a_{32} \quad a_{33}], S = [a_{34}]$$

Thus, A may be regarded as a *super matrix* whose elements are blocks (sub-matrices):

$$A = \begin{bmatrix} P & Q \\ R & S \end{bmatrix}$$

In other words, a matrix partitioned into sub-matrices, or blocks, is called a *partitioned matrix*. Quite naturally, the partitioning of a matrix may be done in a variety of ways. A special case of partitioned matrices are the *bordered matrices*, and is of the form:

$$A_n = \left\|\begin{array}{|c|c|} \hline A_{n-1} & G_n \\ \hline H_n & a_{nn} \\ \hline \end{array}\right\|$$

where,

$$A_{n-1} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1,n-1} \\ a_{21} & a_{22} & \dots & a_{2,n-1} \\ \dots & \dots & \dots & \dots \\ a_{n-1,1} & a_{n-1,2} & \dots & a_{n-1,n-1} \end{bmatrix}$$

is a matrix of order $(n-1)$;

$$G_n = \begin{bmatrix} a_{1,n} \\ a_{2,n} \\ \dots \\ a_{n-1,n} \end{bmatrix} \text{ is a column matrix;}$$

and

$$H_n = \begin{bmatrix} a_{n,1} & a_{n,2} & \dots & a_{n,n-1} \end{bmatrix}$$

is a row matrix.

Another important special case of partitioned matrices are the *quasi-diagonal matrices* and is of the form:

$$A = \begin{bmatrix} [A_1] & & \\ & [\ddots] & \\ & & [A_r] \end{bmatrix}$$

where the blocks $A_k (k = 1, 2, ..., r)$ are square matrices of (generally speaking) different orders, all other elements being zeros.

We have that,

$$\det(A) = \det(A_1)...\det(A_r) \tag{10.13.1}$$

Partitioned matrices are convenient in that operations involving them are carried out formally by the same rules used for ordinary matrices.

However, in as much as many of the operations in the ordinary matrices are feasible in partitioned matrices, we shall be restricting ourselves to only algebraic operations with respect to addition and multiplication of partitioned matrices in this section.

10.13.1 Addition of Partitioned Matrices

Let $A = [A_{ij}]$ and $B = [B_{ij}]$ be partitioned matrices, such that;

$$A = \begin{bmatrix} A_{11} & A_{12} & ... & A_{1q} \\ \vdots & ... & ... & ... \\ A_{p1} & A_{p2} & ... & A_{pq} \end{bmatrix} \tag{10.13.2}$$

and

$$B = \begin{bmatrix} B_{11} & B_{12} & ... & B_{1s} \\ \vdots & ... & ... & ... \\ B_{r1} & B_{12} & ... & B_{rs} \end{bmatrix} \tag{10.13.3}$$

are conformal, that is; $p = r, \quad q = s$ and the blocks A_{ij} and B_{ij} have the same order, then;

$$A + B = \begin{bmatrix} A_{11} + B_{11} & A_{12} + B_{12} & ... & A_{1q} + B_{1q} \\ \vdots & ... & ... & ... \\ A_{p1} + B_{p1} & A_{p2} + B_{p2} & ... & A_{pq} + B_{pq} \end{bmatrix} \tag{10.13.4}$$

Indeed, in order to add the matrices A and B it is necessary to add the corresponding elements, but it is obvious that the same result is achieved if we add the corresponding blocks (sub-matrices) of these matrices.

Remark 30 *Subtraction of matrices is performed analogously.*

10.13.2 Multiplication of Partitioned Matrices

Suppose the partitioned matrices A and B have the structure given in (10.13.2) and (10.13.3), respectively, and that $q = r$. Assume that all the blocks A_{ij} and $B_{jk}(i = 1, 2, ..., p), (j = 1, 2, ..., q), (k = 1, 2, ..., s)$ are such that the number of columns of block A_{ij} is equal to the number of rows of block B_{jk}. In the special case when all blocks A_{ij} and B_{ij} are square and have the same order, this assumption is definitely fulfilled. Then we say that the blocks are conformal for multiplication and the products of the matrices A and B is the partitioned matrix:

$$C = \begin{bmatrix} C_{11} & C_{12} & ... & C_{1s} \\ C_{21} & C_{22} & ... & C_{2s} \\ ... & ... & ... & ... \\ C_{p1} & C_{p2} & ... & C_{ps} \end{bmatrix} \qquad (10.13.5)$$

where,

$$\begin{aligned} C_{ik} &= A_{i1}B_{1k} + A_{i2}B_{2k} + ... + A_{iq}B_{qk} \\ &= \sum_{j=1}^{q} A_{ij}B_{jk}, \quad (i = 1, 2, ..., p; k = 1, 2, ..., s) \end{aligned} \qquad (10.13.6)$$

that is, the matrices A and B are multiplied together as if the blocks (sub-matrices) were numbers.

Example 10.13.1

Find the product of the partitioned matrices:

$$A = \begin{bmatrix} -1 & 2 & \vdots & 0 \\ 1 & 3 & \vdots & -2 \\ ... & ... & \vdots & ... \\ 1 & 0 & \vdots & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -1 & 2 & \vdots & 0 \\ 2 & 1 & 3 & \vdots & 1 \\ ... & ... & ... & \vdots & ... \\ -3 & 2 & 0 & \vdots & 4 \end{bmatrix}$$

using block multiplication.

Solution

The sub-matrices are:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} -1 & 2 & \vdots & 0 \\ 1 & 3 & \vdots & -2 \\ ... & ... & \vdots & ... \\ 1 & 0 & \vdots & 1 \end{bmatrix}$$

and

$$B \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} 1 & -1 & 2 & \vdots & 0 \\ 2 & 1 & 3 & \vdots & 1 \\ ... & ... & ... & \vdots & ... \\ -3 & 2 & 0 & \vdots & 4 \end{bmatrix}$$

Thus,

$$\begin{aligned} AB &= \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{bmatrix} \\ &= \begin{bmatrix} \begin{bmatrix} -1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 3 \end{bmatrix} \\ \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 3 \end{bmatrix} \\ \\ + \begin{bmatrix} 0 \\ -2 \end{bmatrix} \begin{bmatrix} -3 & 2 & 0 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ -2 \end{bmatrix} [4] \\ +[1] \begin{bmatrix} -3 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + [1][4] \end{bmatrix} \end{aligned}$$

$$AB = \begin{bmatrix} \begin{bmatrix} 3 & 3 & 4 \\ 7 & 2 & 11 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 6 & -4 & 0 \end{bmatrix} & \begin{bmatrix} 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 0 \\ -8 \end{bmatrix} \\ \begin{bmatrix} 1 & -1 & 2 \end{bmatrix} + \begin{bmatrix} -3 & 2 & 0 \end{bmatrix} & [0] + [4] \end{bmatrix}$$

$$= \begin{bmatrix} \begin{bmatrix} 3 & 3 & 4 \\ 13 & -2 & 11 \end{bmatrix} & \begin{bmatrix} 2 \\ -5 \end{bmatrix} \\ \begin{bmatrix} -2 & 1 & 2 \end{bmatrix} & [4] \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 3 & 4 & 2 \\ 13 & -2 & 11 & -5 \\ -2 & 1 & 2 & 4 \end{bmatrix}$$

You may cross-check the result with direct multiplication to see that it is the same.

10.14 Systems of Linear Equations

As mentioned earlier in the beginning of this chapter, the theory of linear equations plays a crucial role in the subject of linear algebra. Indeed, the focal point of linear algebra is the solution of linear equations, since many problems in linear algebra are equivalent to studying a system of linear equations.

A simple example of a system (or a set) is simultaneous linear equations given by two adjacent equations:

$$\begin{aligned} a_1 x + b_1 y &= C_1 \\ a_2 x + b_2 y &= C_2 \end{aligned}$$

where, $a_1, a_2;\ b_1, b_2;\ C_1$ and C_2 are given constants or numbers. A solution of such system of equations is a pair of numbers $(\bar{x}, \bar{y})$ such that when x is replaced by $\bar{x}$ and y by $\bar{y}$, both equations are satisfied simultaneously; i.e., both became true propositions on equality. Generalizing this situation, consider the following system of m linear equations in n unknown:

$$\begin{array}{rcl} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n & = & b_1 \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n & = & b_2 \\ \dots\dots\dots\dots\dots\dots\dots & ... & ... \\ a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n & = & b_m \end{array} \qquad (10.14.1)$$

This system of equation can be expressed in a more compact form either as;

$$\sum_{j=1}^{n} a_{ij}x_j = b_i, \quad (i = 1, 2, ..., m) \qquad (10.14.2)$$

or by using a matrix equation in the form;

$$\begin{bmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ ... & ... & ... & ... \\ a_{m1} & a_{m2} & ... & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \qquad (10.14.3)$$

Thus, if we put $A = [a_{ij}]_{m,n}$; $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ and $B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$,

then we can write the system of linear equations in matrix form as:

$$AX = B \qquad (10.14.4)$$

where A is called the matrix of the coefficients, X the matrix of variables and B is the matrix of constants (real numbers). The system is said to be *homogeneous* if the matrix of the constants i.e. B is equal to a null or zero matrix.
i.e,

$$AX = 0 \qquad (10.14.5)$$

Otherwise it is *non-homogeneous* when not all b_i's are equal to zero

The solution set n of

$$X = (y_1, y_2, ..., y_n) \qquad (10.14.6)$$

of real numbers is called the *general solution.*

Methods of solving systems of linear equations are subdivided into two groups:

(i) Exact method: which are finite algorithms for computing the solution or roots of the system (such, as the, inverse method, the Cramer's rule, the Gaussian elimination method.

(ii) Iterative method: which permit obtaining the roots or solutions of a system to a given accuracy by means of convergent infinite processes (these include the Jacobian iterative, and the Gauss Seidel iteration). However, because of unavoidable rounding, the results of even the exact methods are approximate, and error estimates of the solutions are, in the general case, involved. Even, in the case of the iterative processes, we have the compounded error of method. In this text, we shall consider only the exact methods; the reader who wishes to read further can pick up any numerical analysis text for the iterative methods.

10.14.1 Solvability of Systems of Linear Equations

The investigation of the solutions of a system of linear equations leads to three problems which have to be treated each on its own merit:

(a) The first is the question of existence of solutions - this asks under what conditions a system has solutions; for even the single equation: $0.x = 1$ has no solution.

(b) The second problem is to find a method that gives a solution of a given system of linear equations.

(c) Finally, the third problem is to describe the totality of all solutions of the given system of equations.

Consequently, a system of linear equations is said to be *consistent or solvable* if it has at least one solution. Conditions for consistency or solvability of linear equations where they exist is be discussed below.

Suppose, we consider the linear equation, $ax = b$. There are three possibilities:

(i) If $a \neq 0$, then there is a unique solution given by $x = a^{-1}b$

(ii) If $a = b = 0$, then for every real number $x, 0 \times x = 0$; thus, we have infinite number of solutions.

(iii) If $a = 0$ and $b \neq 0$, then the equation is inconsistent and not solvable, i.e. solution does not exist.

If we now have a system of equation; $AX = B$, if A is a square, i.e. the case where the number of equations is equal to the number of unknowns, then (i) to (iii) are possible. However, if A is rectangular, i.e. the case where the number of equations is not equal to the number of unknowns, only (ii) and (iii) are possible.

In short, any homogeneous system of linear equations i.e. $AX = 0$ is always consistent and solvable since it always has at least one solution, namely;

$$x_1 = 0, ... = x_n$$

This solution is called the *trivial* solution, while other solution of the homogeneous system is called a *non-trivial* solution. On the other hand, a non-homogeneous system of linear equations i.e. $AX = B$, (where, not all the b_i are equal to zero), could have no solution as in the case of (iii) above, in which case the system is inconsistent and therefore not solvable.

10.14.2 Solution by Inversion of Matrices

Consider the system of n linear equations in n unknowns in (10.14.1);

$$\left.\begin{array}{lcl} a_{11}x_1 + a_{123}x_2 + ... + a_{1n}x_n & = & b_1, \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n & = & b_2, \\ .... & = & ..... \\ a_{n1}x_1 + a_{n2}x_2 + ... + a_{nn}x_n & = & b_n \end{array}\right\}$$

Let the matrix of coefficients of (10.14.1) be denoted by;

$$A = \begin{bmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ ... & ... & ... & ... \\ a_{n1} & a_{n2} & ... & a_{nn} \end{bmatrix} \qquad (10.14.7)$$

the column of its constant terms by;

$$B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \tag{10.14.8}$$

and the column of the unknowns (variables) by;

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \tag{10.14.9}$$

then the system (10.14.1) can be written as in (10.14.4);

$$AX = B$$

in the form of a matrix equation.

The set of variables $x_1, x_2, ..., x_n$ which reduce (10.14.1) to an identity is called the *solution set* of the system, and the variables x_i; $(i = 1, 2, \ldots, n)$ are termed the *roots* of the system.

If the matrix of the coefficients A is non-singular, that is,

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{vmatrix} \tag{10.14.10}$$

then (10.14.1) or the matrix equation (10.14.4) equivalent to it, has a unique solution.

Indeed, if A is invertible, then there is an inverse matrix A^{-1}, such that if we pre-multiply both members of (10.14.4) by the matrix A^{-1}, we get,

$$A^{-1}AX = A^{-1}B$$

or

$$X = A^{-1}B \tag{10.14.11}$$

The expression (10.14.11) obviously yields a solution of (10.14.4) and since every solution is of the form (10.14.11), the solution is therefore unique.

Example 10.14.1

Solve the following system of equations using inverse method:

$$\text{(i)}\quad \begin{array}{rcrcr} 3x & + & 2y & = & 12 \\ 4x & - & 3y & = & -1 \end{array}$$

$$\text{(ii)}\quad \begin{array}{rcrcrcr} x_1 & + & x_2 & + & x_3 & = & 1 \\ 2x_1 & + & 2x_2 & + & x_3 & = & 4 \\ 3x_1 & - & x_2 & + & 2x_3 & = & -3 \end{array}$$

Solution

(i) Write the system in matrix form:

$$\begin{bmatrix} 3 & 2 \\ 4 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 12 \\ -1 \end{bmatrix}, \text{ i.e. } AX = B$$

The determinant of matrix A of the given system is

$$\det(A) = \begin{vmatrix} 3 & 2 \\ 4 & -3 \end{vmatrix} = -17 \neq 0$$

Hence A is invertible, and computing the inverse matrix A^{-1}, we obtain;

$$A^{-1} = -\frac{1}{17} \begin{bmatrix} -3 & -2 \\ -4 & 3 \end{bmatrix}$$

So that,

$$\begin{bmatrix} x \\ y \end{bmatrix} = -\frac{1}{17} \begin{bmatrix} -3 & -2 \\ -4 & 3 \end{bmatrix} \begin{bmatrix} 12 \\ -1 \end{bmatrix} = -\frac{1}{17} \begin{bmatrix} -34 \\ -51 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

Thus, $x = 2, \quad y = 3$.

(ii) Writing the system in matrix form, we have;

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 1 \\ 3 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ -3 \end{bmatrix} \text{ i.e. } AX = B$$

The determinant, A of the given system is

$$\det(A) = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & 1 \\ 3 & -1 & 2 \end{vmatrix} = -4 \neq 0$$

Hence A is invertible, and computing the inverse matrix A^{-1}, we obtain;

$$A^{-1} = -\frac{1}{4}\begin{bmatrix} 5 & -3 & -1 \\ -1 & -1 & 1 \\ 8 & 4 & 0 \end{bmatrix}$$

so that,

$$\begin{aligned} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= -\tfrac{1}{4}\begin{bmatrix} 5 & -3 & -1 \\ -1 & -1 & 1 \\ 8 & 4 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 4 \\ -3 \end{bmatrix} \\ &= -\tfrac{1}{4}\begin{bmatrix} -4 \\ -8 \\ 8 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} \end{aligned}$$

$$\therefore \qquad x_1 = 1,\ x_2 = 2,\ x_3 = -2$$

Remark 31 *The computation of the inverse A^{-1} of a matrix A of order $n > 4$ directly is very cumbersome and hence the inverse method is seldom used for practical purposes. On the other hand, if A^{-1} does not exist (i.e. if A is singular), then the system $AX = B$ either has infinite solutions or no solution at all, depending on whether the system is consistent or not.*

However, one may not necessarily compute the inverse of A, i.e. A^{-1} in order to obtain a unique solution of a given system of equation which is solvable, as can be shown in the next method, known as Cramer's rule.

10.14.3 Solution by Cramer's Rule

Using expression (10.14.10) above, i.e. $X = A^{-1}B$, it is easy to obtain formula for the unknowns of the system (10.14.1). Recall that;

$$A^{-1} = \frac{1}{\triangle} A^* \tag{10.10.8}$$

where

$$A^* = \begin{bmatrix} A_{11} & A_{21} & ... & A_{n1} \\ \vdots & ... & ... & ... \\ A_{1n} & A_{2n} & ... & A_{nn} \end{bmatrix}$$

is the adjoint of $A(A_{ij}$ are the cofactors of the elements $a_{ij})$.
Therefore,

$$X = \frac{1}{\triangle} A^* B \tag{10.14.11}$$

i.e.,

$$X = \frac{1}{\triangle} \begin{bmatrix} b_1 A_{11} + b_2 A_{21} + & ... & +b_n A_{n1} \\ .... & ... & ... \\ b_1 A_{1n} + b_2 A_{2n} + & ... & b_n A_{nn} \end{bmatrix} \tag{10.14.12}$$

or

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \frac{1}{\triangle} \begin{bmatrix} \triangle_1 \\ \triangle_2 \\ \vdots \\ \triangle_n \end{bmatrix} \tag{10.14.13}$$

where

$$\triangle_i = b_1 A_{1i} + b_2 A_{2i} + ... + b_n A_{ni}, (i = 1, 2, ..., n) \tag{10.14.14}$$

and since

$$a_{1i} A_{1i} + a_{2i} A_{2i} + ... + a_{ni} A_{ni} = \triangle \tag{10.14.15}$$

by definition, it follows that if we replace the i-th column

$$\begin{bmatrix} a_{1i} \\ \vdots \\ a_{ni} \end{bmatrix} \text{ of } A \text{ by } B = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

we obtain $\triangle_i$ and hence

$$x_i = \frac{\triangle_i}{\triangle} \qquad (10.14.16)$$

Equation (10.14.16) now becomes *Cramer's Rule*, thus,

$$x_1 = \frac{\triangle_1}{\triangle}, x_2 = \frac{\triangle_2}{\triangle}, ..., x_n = \frac{\triangle_n}{\triangle} \qquad (10.14.17)$$

Therefore, if the determinant of system (10.14.1) is non-zero, $\triangle \neq 0$, then the system is non-singular, and hence has a unique solution X defined by the matrix formula (10.14.10) or by the scalar formulas (10.14.16) equivalently.

Example 10.14.3

Using Cramer's rule to solve:

$$\begin{array}{rcrcrcr} & & x & + & 4z & = & -2 \\ 2x & + & y & + & 2z & = & 1 \\ 3x & - & y & + & 2z & = & 1 \end{array}$$

Solution

Representing the system in matrix form:

$$\begin{bmatrix} 1 & 0 & 4 \\ 2 & 1 & 2 \\ 3 & -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \text{ i.e. } AX = B$$

The determinant of matrix, A of the given system is,

$$\det(A) = \triangle = \begin{vmatrix} 1 & 0 & 4 \\ 2 & 1 & 2 \\ 3 & -1 & 2 \end{vmatrix} = -16$$

Computing the supplementary determinants, we obtain;

$$\triangle_1 = \left| \begin{array}{rrr} -2 & 0 & 4 \\ 1 & 1 & 2 \\ 1 & -1 & 2 \end{array} \right] = -16$$

$$\triangle_2 = \left| \begin{array}{lrr} 1 & -2 & 4 \\ 2 & 1 & 2 \\ 3 & 1 & 2 \end{array} \right| = -8$$

$$\triangle_3 = \left| \begin{array}{lrr} 1 & 0 & -2 \\ 2 & 1 & 1 \\ 3 & -1 & 1 \end{array} \right| = 12$$

where

$$\begin{aligned} x &= \frac{\triangle_1}{\triangle} = \frac{-16}{-16} = 1 \\ y &= \frac{\triangle_2}{\triangle} = \frac{-8}{-16} = \frac{1}{2} \\ z &= \frac{\triangle_3}{\triangle} = \frac{12}{-16} = -\frac{3}{4} \end{aligned}$$

Thus, the solution of a linear system (10.14.1) in n unknowns reduces to evaluating the $(n+1)$th determinant of order n. In other words, if n is large, evaluating the determinants is a tiresome operation. Hence, direct methods have been elaborated for finding the roots of a linear system of equations.

10.14.4 The Gaussian Elimination Method

One of the oldest and most commonly used technique for finding the solution of a system of linear equation is via an algorithm for the successive elimination of the unknowns. This method is called *Gaussian elimination.* Gaussian Elimination is particularly suitable for computer program solution and for this reason it has gained its significance in recent years. The algorithm is made up of two parts, the first part consists of a step-by-step triangularization of the system, so that it is finally put into triangular form; the second part consists of solving the triangular system by back-substitution.

Consider the general system of n equation in n unknowns;

$$\left.\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & ... & + & a_{1n}x_n & = & a_{1,n+1} \\ a_{21}x_1 & + & a_{22}x_2 & + & ... & + & a_{2n}x_n & = & a_{2,n+1} \\ ... & & ... & & ... & & ... & = & ... \\ a_{n1}x_1 & + & a_{n2}x_2 & + & ... & + & a_{nn}x_n & = & a_{n,n+1} \end{array}\right\} \tag{10.14.18}$$

where $a_{i,n+1}(i = 1, 2, ...n)$ are the constants.

The system (10.14.18) is represented in matrix form as:

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{1,n+1} \\ a_{a,n+1} \\ \vdots \\ a_{n,n+1} \end{bmatrix} \tag{10.14.19}$$

or more compactly as

$$AX = Y$$

in the form of a matrix equation.

We start with the $n \times n + 1$ rectangular matrix $(A : Y)$ whose first n columns are the columns of A, and whose $(n + 1)^{th}$ column is Y i.e. the constants. That is,

$$(A : Y) = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & : & a_{1,n+1} \\ a_{21} & a_{22} & \dots & a_{2n} & : & a_{2,n+1} \\ \dots & \dots & \dots & \dots & : & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} & : & a_{n,n+1} \end{bmatrix} = M \tag{10.14.20}$$

The matrix in (10.14.20) is called the *augmented matrix* of A. Now, we wish to transform this augmented matrix with a sequence of the usual elementary row operations to an equivalent triangular matrix.

In the first stage of the algorithm, let the leading or pivot element $a_{11} \neq 0$. Clearly, the algorithm cannot work if the pivot element is zero. In such a case, one must interchange rows so that the pivot is not zero. However, since the pivot element is not zero, dividing the elements of the first or principal row of (10.14.20) by a_{11}, we get the first equation as;

$$x_1 + b_{12}x_2 + ... + b_{1n}x_n = b_{1,n+1} \tag{10.14.21}$$

and the augmented matrix becomes;

$$\begin{bmatrix} 1 & \frac{a_{12}}{a_{11}} & ... & \frac{a_{1n}}{a_{11}} & : & \frac{a_{1,n+1}}{a_{11}} \\ a_{21} & a_{22} & ... & a_{2n} & : & a_{2,n+1} \\ ... & ... & ... & ... & : & ... \\ a_{n1} & a_{n2} & ... & a_{nn} & : & a_{n,n+1} \end{bmatrix} \tag{10.14.22}$$

For simplicity, we can write,

$$b_{1j} = \frac{a_{1j}}{a_{11}}, \quad (j = 2, 3, ..., n+1) \tag{10.14.23}$$

Now, using principal row of (10.14.22), it is easy to eliminate all the elements in the first column of (10.14.22) below the pivot. To do this, subtract Row1 multiplied by a_{21} from Row2 of (4), subtract Row1 multiplied by a_{31} from Row3 of (10.14.22) and so on. We finally get a matrix $M^{(1)}$ consisting of $(n-1)$ rows X_n columns:

$$\begin{bmatrix} a_{22}(1) & ... & a_{2n}^{(1)} & : & a_{2,n+1}^{(1)} \\ ... & ... & ... & : & ... \\ a_{n1}^{(1)} & ... & a_{nn}^{(1)} & : & a_{n,n+1}^{(1)} \end{bmatrix} \tag{10.14.24}$$

where the elements $a_{ij}^{(1)}(i, j \geq 2)$ are computed from the formula:

$$a_{ij}^{(1)} = a_{ij} - a_{i1}b_{1j} \quad (i, j \geq 2) \tag{10.14.25}$$

Now,dividing the elements of the first or principal row of (10.14.24) by the pivot element, $a_{22}^{(1)} \neq 0$, we get the second equation as;

$$x_2 + ... + b_{2n}^{(1)}x_n = b_{2,n+1}^{(1)} \tag{10.14.26}$$

and the augmented matrix becomes;

$$\begin{bmatrix} 1 & ... & \frac{a_{2n}^{(1)}}{a_{22}^{(1)}} & : & \frac{a_{2,n+1}^{(1)}}{a_{22}^{(1)}} \\ ... & ... & ... & ... & ... \\ a_{n2}^{(1)} & ... & a_{nn}^{(1)} & : & a_{n,n+1}^{(1)} \end{bmatrix} \tag{10.14.27}$$

Similarly, we can write;

$$b_{2j}^{(1)} = \frac{a_{2j}^{(1)}}{a_{22}^{(1)}};\ (j = 3, 4...n+1) \tag{10.14.28}$$

Now, eliminating the elements of the first column in (10.14.27) below the pivot in the same way that we did in (10.14.22), and continuing in this manner after $(n-1)$ repetitions of the process, the augmented matrix (10.14.20) is reduced to the triangular form:

$$B = \begin{bmatrix} 1 & b_{12} & ... & b_{1n} & : & b_{1,n+1} \\ 0 & 1 & ... & b_{2n}^{(1)} & : & b_{2,n+1}^{(1)} \\ ... & ... & ... & ... & ... & ... \\ 0 & 0 & ... & 1 & : & b_{n,n+1}^{(n-1)} \end{bmatrix} \tag{10.14.29}$$

Consequently, the original system (10.14.18) is reduced to the triangular system:

$$\left.\begin{array}{cccccccc} x_1 & + & b_{12}x_2 & + & ... & + & b_{1n}x_n & = & b_{1,n+1} \\ & & x_2 & + & ... & + & b_{2n}^{(1)}x_n & = & b_{2,n+1}^{(1)} \\ & & ... & ... & ... & ... & ... & ... & ... \\ & & & & & & x_n & = & b_{n,n+1}^{(n-1)} \end{array}\right\} \tag{10.14.30}$$

where

$$x_n = \frac{a_{n,n+1}^{(n-1)}}{a_{nn}^{(n-1)}} = b_{n,n+1}^{(n-1)} \tag{10.14.31}$$

The remaining unknowns are successively determined by *back-substitution*. Thus, the system of solving a linear system of equations by the Gaussian Elimination method reduces to the construction of an equivalent system (10.14.30) having a triangular system. A necessary and sufficient condition for using this method is that all the leading or pivot elements be non-zero and always the upper left element of the corresponding matrix. The process of finding the coefficients $b_{ij}^{(j-1)}$ of a triangular system may be called *forward substitution* (direct procedure), while, the process of finding the values of the unknowns is called the *back - substitution* (reverse procedure).

Remark 32 *Generally, one may apply the triangularization process to a system of linear equations without first calculating the determinant of coefficients,* $|A|$. *When this approach is taken, it may be observed that at some stage of the process, a degenerate equation, of the form*

$$0x_1 + 0x_2 + ... + 0x_n = b \tag{10.14.31}$$

would be obtained, signaling that $\det(A) = 0$. *In such case, the system either has no solution, and is said to be inconsistent (i.e. not solvable) or has an infinite number of solutions if* $b = 0$ *(i.e. homogeneous system). However, we note that a system is inconsistent only if some degenerate equation has a non-zero constant.*

Example 10.14.3

Use the Gaussian elimination process to solve the system:

$$\begin{aligned} 2x_1 + 4x_2 - 4x_3 &= 16 & (i) \\ 2x_1 - 4x_2 + 2x_3 &= 0 & (ii) \\ x_1 - 2x_2 - x_3 &= 8 & (iii) \end{aligned}$$

Solution

Step I

Since the coefficient of x_1 in Equation (iii) is a unit; for simplicity, we interchange Equations (i) and (iii) i.e.,

$$\begin{aligned} x_1 - 2x_2 - x_3 &= 8 & (iii) \\ 2x_1 - 4x_2 + 2x_3 &= 0 & (ii) \\ 2x_1 + 4x_2 - 4x_3 &= 16 & (i) \end{aligned}$$

Step II

Eliminate x_1 from Equations (ii) and (i)
Equation (iii)$\times(-2)+$ Eqn. (ii)
Equation (iii) $\times(-2)+$ Equation (i):

$$\begin{aligned} x_1 - 2x_2 - x_3 &= 8 \\ 4x_3 &= -16 & (iv) \\ 8x_2 - 2x_3 &= 0 & (v) \end{aligned}$$

Step III

Since the coefficient of x_2 in Equation (iv) is zero and that of Equation (v) is non-zero, we therefore interchange Equations (iv) and

(v) such that;

$$\begin{aligned} x_1 - 2x_2 - x_3 &= 8 \\ 8x_2 - 2x_3 &= 0 \\ 4x_3 &= -16 \end{aligned}$$

Step IV

We now use back substitution to solve for x_1, x_2 and x_3:

$$\begin{aligned} 4x_3 &= -16 \\ x_3 &= -4 \end{aligned}$$

Substitute for x_3 in (V):

$$\begin{aligned} 8x_2 - 2(-4) &= 0 \\ 8x_2 &= -8 \\ x_2 &= -1 \end{aligned}$$

Substitute for x_3 and x_2 in (i):

$$\begin{aligned} x_1 - 2(-1) - (-4) &= 8 \\ x_1 &= 8 - 2 - 4 \\ x_1 &= 2 \end{aligned}$$

$\therefore \quad x_1 = 2, \quad x_2 = -1$ and $x_3 = -4$

Alternatively, we can form the augmented matrix of the system:

$$AX = \begin{bmatrix} 2 & 4 & -4 \\ 2 & -4 & 2 \\ 1 & -2 & -1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}; \quad B = \begin{bmatrix} 16 \\ 0 \\ 8 \end{bmatrix}$$

and solve using the same elementary row operations i.e.,

$$A : B = \begin{bmatrix} 2 & 4 & -4 & : & 16 \\ 2 & -4 & 2 & : & 0 \\ 1 & -2 & -1 & : & 8 \end{bmatrix} R_1 \leftrightarrow R_3$$

$$= \begin{bmatrix} 1 & -2 & -1 & : & 8 \\ 2 & -4 & 2 & : & 0 \\ 2 & 4 & -4 & : & 16 \end{bmatrix} \begin{array}{l} R_2 \to -2R_1 + R_2 \\ R_3 \to -2R_1 + R_3 \end{array}$$

$$= \begin{bmatrix} 1 & -2 & -1 & 8 \\ 0 & 0 & 4 & -16 \\ 0 & 8 & -2 & 0 \end{bmatrix} R_2 \leftrightarrow R_3$$

$$= \begin{bmatrix} 1 & -2 & -1 & : & 8 \\ 0 & 8 & -2 & : & 0 \\ 0 & 0 & 4 & : & -16 \end{bmatrix}$$

Writing out the final equation, we have;

$$\begin{aligned} x_1 - 2x_2 - x_3 &= 8 \\ 8x_2 - 2x_3 &= 0 \\ 4x_3 &= -16 \end{aligned}$$

which is the same with what we have in step III above and hence the solution follows.

Example 10.14.4

Solve the following non-homogeneous system of linear equations:

$$\left.\begin{array}{rcrcrcrcr} 2x_1 & + & x_2 & - & 5x_3 & + & x_4 & = & 8 \\ & & x_1 & - & 3x_2 & - & 6x_4 & = & 9 \\ & & 2x_2 & - & x_3 & + & 2x_4 & = & -5 \\ x_1 & + & 4x_2 & - & 7x_3 & + & 6x_4 & = & 0 \end{array}\right\} \qquad (i)$$

Solution

The augmented matrix of the system $AX = Y$ is given as:

$$AX = \begin{bmatrix} 2 & 1 & -5 & 1 \\ 1 & -3 & 0 & -6 \\ 0 & 2 & -1 & 2 \\ 1 & 4 & -7 & 6 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix}; \quad Y = \begin{bmatrix} 8 \\ 9 \\ -5 \\ 0 \end{bmatrix} \qquad (ii)$$

is represented thus;

$$(A:Y) = \begin{bmatrix} 2 & 1 & -5 & 1 & : & 8 \\ 1 & -3 & 0 & -6 & : & 9 \\ 0 & 2 & -1 & 2 & : & -5 \\ 1 & 4 & -7 & 6 & : & 0 \end{bmatrix} = M \qquad (iii)$$

where A is the matrix of the coefficients and Y the column matrix of the constants.

Step I

$a_{11} = 2 \neq 0$ and has the largest absolute value among the elements in the first column, so $a_{11} = 2$ becomes the pivot or principal element and dividing through the first row i.e. Row1 by the pivot element, we obtain our first equation as:

$$x_1 + \frac{1}{2}x_2 - \frac{5}{2}x_3 + \frac{1}{2}x_4 = 4 \qquad (iv)$$

and the consequent augmented matrix becomes;

$$\begin{bmatrix} \boxed{1} & \frac{1}{2} & \frac{-5}{2} & \frac{1}{2} & : & 4 \\ 1 & -3 & 0 & -6 & : & 9 \\ 0 & 2 & -1 & 2 & : & -5 \\ 1 & 4 & -7 & 6 & : & 0 \end{bmatrix} \qquad (v)$$

Step II

Eliminate all the elements below the pivot in column 1 of (v) and hence compute the elements $a_{ij}^{(1)}$; $(i, j \geq 2)$. Thus, from the formula;

$$a_{ij}^{(1)} = a_{ij} - b_{i1}b_{1j};\ (i, j \geq 2)$$

where $b_{1j} = \frac{a_{1j}}{a_{11}}$ as follows:

$$\begin{array}{lllllll}
a_{22}^{(1)} & = & a_{22} - a_{21}b_{12} & = & -3 - (1 \times \frac{1}{2}) & = & -\frac{7}{2} \\
a_{23}^{(1)} & = & a_{23} - a_{21}b_{13} & = & 0 - (1 \times \frac{-5}{2}) & = & \frac{5}{2} \\
a_{24}^{(1)} & = & a_{24} - a_{21}b_{24} & = & -6 - (1 \times \frac{1}{2}) & = & -\frac{13}{2} \\
a_{25}^{(1)} & = & a_{25} - a_{21}b_{15} & = & 9 - (1 \times 4) & = & 5
\end{array}$$

$$\begin{array}{lllllll}
a_{32}^{(1)} & = & a_{32} - a_{31}b_{12} & = & 2 - (0) & = & 2 \\
a_{33}^{(1)} & = & a_{33} - a_{31}b_{12} & = & -1 & & \\
a_{34}^{(1)} & = & 2 & & & & \\
a_{35}^{(1)} & = & -5 & & & &
\end{array}$$

$$\begin{array}{lllllll}
a_{42}^{(1)} & = & a_{42} - a_{41}b_{12} & = & 4 - (1 \times \frac{1}{2}) & = & \frac{7}{2} \\
a_{43}^{(1)} & = & a_{43} - a_{41}b_{13} & = & -7 - (1 \times \frac{-5}{2}) & = & -\frac{9}{2} \\
a_{44}^{(1)} & = & a_{44} - a_{41}b_{14} & = & 6 - (1 \times \frac{1}{2}) & = & \frac{11}{2} \\
a_{45}^{(1)} & = & a_{45} - a_{41}b_{15} & = & 0 - (1 \times 4) & = & -4
\end{array}$$

$$M^{(1)} = \begin{bmatrix} a_{22}^{(1)} & a_{23}^{(1)} & a_{24}^{(1)} & : & a_{25}^{(1)} \\ a_{32}^{(1)} & a_{33}^{(1)} & a_{34}^{(1)} & : & a_{35}^{(1)} \\ a_{42}^{(1)} & a_{43}^{(1)} & a_{44}^{(1)} & : & a_{45}^{(1)} \end{bmatrix} = \begin{bmatrix} \frac{-7}{2} & \frac{5}{2} & -\frac{13}{2} & : & 5 \\ 2 & -1 & 2 & : & -5 \\ \frac{7}{2} & \frac{-9}{2} & \frac{11}{2} & : & -4 \end{bmatrix} \qquad (vi)$$

Now, dividing row 1 by the pivot element;

$$a_{22}^{(1)} = \frac{-7}{2} \neq 0$$

, we have our second equation as:

$$x_2 - \frac{5}{7}x_3 + \frac{13}{7}x_4 = -\frac{10}{7} \qquad (vii)$$

and consequently, we have the augmented matrix as:

$$\begin{bmatrix} \boxed{1} & \frac{-5}{7} & \frac{13}{7} & : & \frac{-10}{7} \\ 2 & -1 & 2 & : & -5 \\ \frac{7}{2} & -\frac{9}{2} & \frac{11}{2} & : & -4 \end{bmatrix} \qquad (viii)$$

Similarly, we compute the elements $a_{ij}^{(2)}(i, j \geq 3)$ from the formula;

$$a_{ij}^{(2)} = a_{ij}^{(1)} - a_{i2}^{(1)} b_{2j}^{(1)};\ (i, j \geq 3)$$

where

$$b_{2j}^{(1)} = \frac{a_{2j}^{(1)}}{a_{22}^{(1)}} \quad \text{as follows:}$$

$$\begin{aligned}
a_{33}^{(2)} &= a_{33}^{(1)} - a_{32}^{(1)} b_{23}^{(1)} &&= -1 - (2 \times -\tfrac{5}{7}) &&= \tfrac{3}{7} \\
a_{34}^{(2)} &= a_{34}^{(2)} - a_{32}^{(1)} b_{24}^{(1)} &&= 2 - (2 \times \tfrac{13}{7}) &&= \tfrac{-12}{7} \\
a_{35}^{(2)} &= a_{35}^{(1)} - a_{32}^{(1)} b_{25}^{(1)} &&= -5 - (2 \times -\tfrac{10}{7}) &&= -\tfrac{15}{7} \\
a_{43}^{(2)} &= a_{43}^{(1)} - a_{42}^{(1)} b_{23}^{(1)} &&= -\tfrac{9}{2} - (\tfrac{7}{2} \times \tfrac{-5}{7}) &&= -2 \\
a_{44}^{(2)} &= a_{44}^{(1)} - a_{42}^{(1)} b_{24}^{(1)} &&= \tfrac{11}{2} - (\tfrac{7}{2} \times \tfrac{13}{7}) &&= -1 \\
a_{45}^{(2)} &= a_{45}^{(1)} - a_{42}^{(1)} b_{25}^{(1)} &&= -4 - (\tfrac{7}{2} \times \tfrac{-10}{7}) &&= 1
\end{aligned}$$

i.e.,

$$M^{(2)} = \begin{bmatrix} a_{33}^{(2)} & a_{34}^{(2)} & : & a_{35}^{(2)} \\ a_{43}^{(2)} & a_{44}^{(2)} & : & a_{45}^{(2)} \end{bmatrix} = \begin{bmatrix} \frac{3}{7} & -\frac{12}{7} & : & -\frac{15}{7} \\ -2 & -1 & : & 1 \end{bmatrix} \qquad \text{(ix)}$$

Dividing row 1 of (ix) by the pivot element:

$$a_{33}^{(2)} = \frac{3}{7} \neq 0$$

we obtain our third equation as:

$$x_3 - 4x_4 = -5 \qquad \text{(x)}$$

and we have the augmented matrix as:

$$\begin{bmatrix} \boxed{1} & -4 & : & -5 \\ -2 & -1 & : & 1 \end{bmatrix} \qquad \text{(xi)}$$

We now obtain our $a_{ij}^{(3)};\ (i, j \geq 4)$ as follows:

$$a_{ij}^{(3)} = a_{ij}^{(2)} - a_{i3}^{(2)} b_{3j}^{(2)} \quad (i, j \geq 4)$$

where

$$\begin{aligned} b_{3j}^{(2)} &= \tfrac{a_{3j}^{(2)}}{a_{33}^{(2)}} \\ a_{44}^{(3)} &= a_{44}^{(2)} - a_{43}^{(2)} b_{34}^{(2)} = -1 - (-2 \times -4) = -9 \\ a_{45}^{(3)} &= a_{45}^{(2)} - a_{43}^{(2)} b_{35}^{(2)} = 1 - (-2 \times -5) = -9 \end{aligned}$$

i.e.,

$$M^{(3)} = [a_{44}^{(3)} : a_{45}^{(2)}] = [-9 : -9] \qquad \text{(xii)}$$

where,

$$x_4 = \frac{a_{45}^{(3)}}{a_{44}^{(3)}} = b_{45}^{(3)}$$

i.e.

$$x_4 = 1 \qquad \text{(xiii)}$$

which becomes our fourth equation. We can now compute the values of the remaining variables using back-substitution from the triangular system of equations (iv), (vii), (x) and (xiii) as follows:

$$\left.\begin{aligned} x_1 + \tfrac{1}{2}x_2 - \tfrac{5}{2}x_3 + \tfrac{1}{2}x_4 &= 4 \\ x_2 - \tfrac{5}{7}x_3 + \tfrac{13}{7}x_4 &= -\tfrac{10}{7} \\ x_3 - 4x_4 &= -5 \\ x_4 &= 1 \end{aligned}\right\} \qquad \text{(xiv)}$$

i.e., from (x) we have;

$$\begin{aligned} x_3 - 4(1) &= -5 \\ \therefore \quad x_3 &= -1 \end{aligned}$$

Similarly, from (vii), we obtain x_2 as;

$$\left.\begin{aligned} x_2 - \tfrac{5}{7}x_3 + \tfrac{13}{7}x_4 &= \tfrac{-10}{7} \\ x_2 - \tfrac{5}{7}(-1) + \tfrac{13}{7}(1) &= \tfrac{-10}{7} \\ \therefore \quad x_2 &= -4 \end{aligned}\right\}$$

Thus from (iv);

$$x_1 + \frac{1}{2}(-4) - \frac{5}{2}(-1) + \frac{1}{2}(1) = 4$$

$$\therefore \qquad x_1 = 3$$

Hence the solutions are: $x_1 = 3,\ x_2 = -4,\ x_3 = -1$ and $x_4 = 1$.

Chapter 11

Vector Space and Subspace

The discussion on Homogeneous Equations in Chapter 10, shows that the solutions of a homogeneous system of equation forms an example of a set of objects that can be added together or multiplied by numbers without leaving the set. A generalization and abstract definition of these properties leads to the concept of *Vector Space*. It lays a solid foundation for metric space topology and abstract algebra in higher mathematics.

11.1 Vector Space

Let F be a non-empty set such that it is closed under $(+)$ an operation called addition and $(*)$ another operation called multiplication and let there exist $a, b, c \in F$ such that the two operations satisfy the following axioms:

(i) $a + b \in F$

(ii) $a * b \in F$

(iii) $(a + b) + c = a + (b + c)$

(iv) $a + b = b + a$

(v) $\exists\ 0 \in F$ such that, $0 + a = a + 0 = a$

(vi) $(a * b) * c = a * (b * c)$

(vii) $a * (b + c) = a * b + a * c$

(viii) $(a + b) * c = a * c + b * c$

(ix) $\exists\ I \in F$ such that, $I * a = a * I = a$

(x) $a^{-1} * a = 1, \forall\ a^{-1} \in F$; then F is called a *field*

For example:

$\mathbb{R}$ = field of real numbers
$\mathbb{Q}$ = field of rational numbers
$\mathbb{C}$ = field of complex numbers

Therefore, a vector space (often called *linear space*), V over a field F defined above, is a set of elements called vectors, such that for every $v_1, v_2 \in V$ and $\alpha \in \mathbb{R}$;

(i) $v_1 + v_2 \in V$

(ii) $\alpha v_1 \in V$

Thus, the set V is closed under vector addition and scalar multiplication. Observe that, by condition (ii), any vector space must contain a zero vector.

Example 11.1.1

(i) Let F be any field, then;

$$F^n = F \times F \times ... \times F; (n \text{ factors})$$

is a vector space over F if we define the operation of addition and scalar multiplication; such that, for an ordered n-turple of numbers,

$$x = (x_1, x_2, ..., x_n),\ y = (y_1, y_2, ..., y_n) \in F^n$$

- $x + y = (x_1 + y_1, x_2 + y_2, ..., x_n + y_n)$
- $\alpha x = (\alpha x_1, \alpha x_2, ..., \alpha x_n)$ for any $\alpha \in F$

(ii) Consider $M_{n,m}(F)$ to be the set of all $(n \times m)$- matrices with elements from a field F. Then, $M_{n,m}(F)$ is a vector space over the field F if we define addition and scalar multiplication such that for every,

$$A = [a_{ij}]_{n,n}, B = [b_{ij}]_{n,m} \in M_{n,m}(F)$$

- $A + B = [a_{ij} + b_{ij}]_{n,m}$
- $\alpha A = [\alpha a_{ij}]_{n,m}$ for any $\alpha \in F$

(iii) Let $F(x)$ denote the set of all polynomials in x with coefficients from a field F. Then, $F(x)$ is a vector space over the field F if we define the operation of addition and scalar multiplication such that for every,

$$a(x) = a_0 + a_1 x + ... + a_n x^n, \ b(x) = b_0 + b_1 x + ... + b_m x^m \in F(x);$$

$n \geq m$

- $a(x) + b(x) = (a_0 + b_0) + (a_1 + b_1)x + ... + (a_m + b_m)x^m$
$$+ a_{m+1}x^{m+1} + ... + a_n x^n$$
- $\alpha * a(x) = \alpha a_0 + (\alpha a_1)x_1 + ... + (\alpha a_n)x^n$, for any $\alpha \in F$

Remark 33 *Two vectors $x = (x_1, x_2, ..., x_n)$ and $y = (y_1, y_2, ..., y_n)$ are considered equal if and only if their coordinate standing in the same positions coincide, that is, if $x_i = y_i$ for $i = 1, 2, ..., n$.*

Remark 34 *Let V and W be vector spaces. The function $T : V \rightarrow W$ is called a linear transformation of V into W if for every $x, y \in V$ and $\alpha \in R$, we have;*

(i) $T(x + y) = Tx + Ty$

(ii) $T(\alpha x) = \alpha Tx$

Thus, if $L(V, W)$ denote the set of all linear transformations of V into W, then $L(V, W)$ is a vector space.

11.2 Subspace

A subspace, S of a vector space, V is a subset of V which is itself a vector space with respect to the operations of addition and scalar multiplication in V. In other words, a subset S will be a subspace if and only if the sum of any two vectors of S lies in S and the product of any vector of S by a scalar lies in S. Geometrically, a *subspace* is simply a linear subspace (line, plane, e.t.c.) through the origin 0.

For example, the vectors of the form $(0, x_2, 0, x_4)$ constitute a subspace of $V_4(F)$, for any field F. We can verify this by taking any two vectors:

$$\tilde{V_1} = (0, y_2, 0, y_4), \tilde{V_2} = (0, z_2, 0, z_4) \in V_4$$

Then,

$$\tilde{V_1} + \tilde{V_2} = (0, y_2 + z_2, 0, y_4 + z_4) = (0, y, 0, z) \in V_4$$

Theorem 11.2.1. *The set of solution of the linear homogeneous system $A\underline{X} = \underline{0}$ is a subspace of $\mathbb{R}^n$. This subspace is called the null space of A.*

Proof. Let

$$S = \{\underline{X} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} : A\underline{X} = \underline{0}\},$$

so that V_1, V_2 be arbitrary members of S, then,

$$V_1 \in S \text{ implies that } AV_1 = 0$$

and

$$V_2 \in S \text{ implies that } AV_2 = 0$$

Thus, by linear transformation,

$$A(V_1 + V_2) = AV_1 + AV_2 = 0 + 0 = 0$$

$$\therefore \qquad V_1 + V_2 \in S$$

Hence, S is closed under addition.
Next, let $\alpha \in \mathbb{R}$ be a scalar, and $V_1 \in S$, then;

$$A(\alpha V_1) = \alpha(AV_1) = \alpha(0) = 0$$

$\therefore \quad \alpha V_1 \in S$.
Thus, S is closed under scalar multiplication and S is a vector space.
□

Example 11.2.1

What is the null space of the matrix, $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$?

Solution

The solution of $A\underline{X} = \underline{0}$ is the null space of A i.e.,

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}^1$$

(Note that the symbol, $\sim$ is used to indicate similar matrices).
Thus,

$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variable is x_2; so, let $x_2 = \alpha$ be arbitrary, then we have;

$$x_1 + 2\alpha = 0 \text{ or } x_1 = -2\alpha$$

$\therefore \quad$ the null space $= \{\alpha(-2, 1) : \alpha \in \mathbb{R}\}$

Theorem 11.2.2. *The set of all linear combinations of any set of vectors in a space V is a subspace of V.*

Proof. Suppose, the vector $\tilde{X} = x_1, x_2, ..., x_n$ is in the vector space V, then the set of all *linear combinations*:

$$\alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_n x_n; \quad \forall \alpha_i \in \mathbb{R}$$

of the x_i is a subspace. This is because of the identities;

$$\begin{aligned}(\alpha_1 x_1 + \alpha_2 x_2 + \ \ldots \ &+ \alpha_n x_n) + (\alpha_1' x_1 + \alpha_2' x_2 + \ldots + \alpha_n' x_n) \\ &= (\alpha_1 + \alpha_1')x_1 + (\alpha_2 + \alpha_2')x_2 + \ldots + (\alpha_n + \alpha_n')x_n,\end{aligned}$$

and

$$\alpha'(\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n) = (\alpha' \alpha_1)x_1 + (\alpha' \alpha_2)x_2 + \ldots + (\alpha' \alpha_n)x_n,$$

which is valid for all vectors x_i and for all scalars α_i, α_i', and α'. Which completes the proof. □

Remark 35 *This subspace in the Theorem 11.2.2 is evidently the smallest subspace containing all the given vectors; hence it is called the subspace generated or spanned by them. The subspace spanned by a single vector $x_1 \neq 0$ is the set S_1 of all scalar multiples αx_1; geometrically, this is simply the line through the origin and x_1. Similarly, the subspace spanned by two non-collinear vectors x_1 and x_2 is the plane passing through the origin, x_1, and x_2.*

Example 11.2.2

In $\mathbb{R}^3$, consider the vector $\tilde{X}_1 = (2, 1, 0)$, $\tilde{X}_2 = (1, 3, 0)$ and $\tilde{X}_3 = (0, 0, 3)$. Show that $\tilde{X}_4 = (1, 2, 9)$ is a linear combination of $\tilde{X}_1, \tilde{X}_2$, and $\tilde{X}_3$.

Solution

If $\tilde{X}_4$ is a linear combination of $\tilde{X}_1, \tilde{X}_2$, and $\tilde{X}_3$, then, there exists scalars $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ such that;

$$\tilde{X}_4 = \alpha_1 \tilde{X}_1 + \alpha_2 \tilde{X}_2 + \alpha_3 \tilde{X}_3$$

so that,

$$(1, 2, 9) = \alpha_1(2, 1, 0) + \alpha_2(1, 3, 0) + \alpha_3(0, 0, 3)$$

Thus, we obtain the following equations:

$$\begin{aligned}1 &= 2\alpha_1 + \alpha_2 \\ 2 &= \alpha_1 + 3\alpha_2 \\ 9 &= 3\alpha_2\end{aligned}$$

and solving we have,

$$\alpha_3 = 3,\ \alpha_2 = \frac{3}{5},\ \alpha_1 = \frac{1}{5}$$

Hence, $\tilde{X}_4$ is a linear combination of $\tilde{X}_1$, $\tilde{X}_2$, and $\tilde{X}_3$.

Theorem 11.2.3. *The intersection, $S \cap T$, of any two subspaces of a vector space V is itself a subspace of V.*

Proof. The intersection of two given subspaces S and T is defined to be the set $S \cap T$ of all those vectors belonging to both S and T. Suppose α and β are two such vectors, then;

(i) $\alpha + \beta \in S$ (Since S is a subspace containing α and β) and likewise
$\alpha + \beta \in T$,
hence,

$$(\alpha + \beta) \in S \cap T$$

(ii) Similarly, any scalar multiple $k \in \mathbb{R}$; $k\alpha \in S$ and $k\alpha \in T$, hence $k\alpha \in S \cap T$.

□

Corollary 11.2.1. *Any two subspaces S and T of a vector space V determine a set $S + T$ consisting of all sums $\alpha + \beta$, for $\alpha \in S$ and $\beta \in T$. By addition and scalar multiplication operation on $S + T$, this set is itself a subspace, and is contained in any other subspace $\mathbb{R}$ containing both S and T. These properties of $S + T$ may be stated thus:*

(i) $S \in S + T, \quad T \in S + T$

(ii) $S \in \mathbb{R}$ *and* $T \in \mathbb{R}$ *imply* $S + T \in \mathbb{R}$

where $S \in \mathbb{R}$ means that the subspace S is contained in the subspace $\mathbb{R}$.

11.3 Linear Dependence and Independence

The vectors, $v_1, v_2, ..., v_m$ in a linear vector space V over a field F are said to be *linearly independent* over F if and only if, for all scalars $\alpha_i \in F$; $(i = 1, 2, .., m)$:

$$\left.\begin{array}{lrcl} & \sum_i^m \alpha_i v_i & = & 0 \\ \text{i.e.} & \alpha_1 v_1 + \alpha_2 v_2 + ... + \alpha_m v_m & = & 0 \\ \text{implies that} & \alpha_1 = \alpha_2 = ... = \alpha_m & = & 0 \end{array}\right\} \quad (11.3.1)$$

Otherwise, the vectors are said to be *linearly dependent.* In other words, if there exists scalars, $\alpha_i \in F$; $(i = 1, 2, ..., m)$, not all zero, such that;

$$\begin{array}{lrcl} & \sum_i^m \alpha_i v_i & = & 0 \\ \text{i.e.} & \alpha_1 v_1 + \alpha_2 v_2 + ... + \alpha_m v_m & = & 0 \end{array}$$

then, the vector space $V(F)$ are termed linearly dependent.

Theorem 11.3.1. *The non-zero vectors, $v_1, v_2, ..., v_m$ in a vector space over a field F, are linearly dependent if and only if one of the vectors V_k is a linear combination of the preceding ones.*

Proof. Whenever the vector V_k is a linear combination of the preceding ones, i.e.,

$$V_k = \alpha_1 v_1 + \alpha_2 v_2 + ... + \alpha_{k-1} v_{k-1}$$

we have a linear relation;

$$\alpha_1 v_1 + \alpha_2 v_2 + ... + \alpha_{k-1} v_{k-1} + (-1) v_k = 0$$

with at least one coefficient (i.e scalar), $-1 \neq 0$. Hence, the vectors are dependent. □

Corollary 11.3.1. *A set of vectors is linearly dependent if and only if it contains a proper (i.e. smaller) subset generating the same subspace.*

Namely, we can delete from the set any one vector which is 0 *or which is a linear combination of the preceding ones, and show that the remaining vectors generate the same subspace. Now, using induction, we obtain: Any finite set of vectors contains a linearly independent subset which spans (generates) the same subspace.*

Example 11.3.1

Let x, y, and z be vectors in a three dimensional vector space, E_3, then the linear dependence of two vectors x and y implies that they are parallel to some straight line; and linear dependence of three vectors x, y, and z means that they are parallel to some plane.

Observe that if some of the vectors are linearly dependent then the whole set of vectors is also linearly dependent.

Suppose, we have a set of vectors;

$$x^{(j)} = (x_1^{(j)}, x_2^{(j)}, ..., x_n^{(j)});\ (j = 1, 2, ..., m)$$

Then, in order to determine the constants, $\alpha_k(k = 1, 2, ..., m)$, we get the following system by virtue of equation (11.3.1):

$$\left.\begin{array}{rcl} \alpha_1 x_1^{(1)} + \alpha_2 x_2^{(2)} + ... + \alpha_m x_m^{(m)} & = & 0 \\ \alpha_1 x_2 + \alpha_2 x_2^{(2)} + ... + \alpha_m x_m^{(m)} & = & 0 \\ \vdots .... & & \\ a_1 x_n^{(1)} + \alpha_2 x_n^{(2)} + ... + \alpha_m x_n^{(m)} & = & 0 \end{array}\right\} \tag{11.3.2}$$

It follows that, the given vectors are linearly dependent if this system has non-trivial solutions, otherwise they are linearly independent.

Consider the matrix of the coordinates:

$$X = \begin{bmatrix} x_1^{(1)} & x_1^{(2)} & ... & x_1^{(m)} \\ x_2^{(1)} & x_2^{(2)} & ... & x_2^{(m)} \\ .... & ... & ... & ... \\ x_n^{(1)} & x_n^{(2)} & ... & x_n^{(m)} \end{bmatrix} \tag{11.3.3}$$

Suppose r is the rank of the matrix, then, the equation (11.3.2) has non-trivial solutions if and only if the rank, $r < m$. Hence, the vectors $x^{(1)}, x^{(2)} ..., x^{(m)}$ are linearly dependent if $r < m$ and linearly independent if $r = m$ (the rank cannot obviously be greater than m). Consequently, the rank of a matrix X gives us the maximum number of linearly independent vectors contained in a given set of vectors.

Thus, if the rank of the matrix X is equal to r, then among the column vectors, $x = x^{(1)}, x^{(2)}, ..., x^{(m)}$:

(i) There will be r linearly independent vectors.

(ii) Every set $r+1$ of vectors $(r+1 \leq m)$ of this collection are linearly dependent. The same is true of row vectors $x_i^{(i)}, ..., x_i^{(m)}$; $(i = 1, 2, ..., n)$ of matrix X.

Example 11.3.2

Show that the following system of vectors are linearly dependent?

$$\begin{aligned} x^{(1)} &= (1, 1, 1, 3) \\ x^{(2)} &= (-1, 0, -5, -6) \\ x^{(3)} &= (1, 2, -1, 2) \\ x^{(4)} &= (-1, 0, 2, 1) \\ x^{(5)} &= (1, 1, -1, 1) \end{aligned}$$

Solution

The matrix of the coordinates are:

$$X = \begin{bmatrix} 1 & -1 & 1 & -1 & 1 \\ 1 & 0 & 2 & 0 & 1 \\ 1 & -5 & -1 & 2 & -1 \\ 3 & -6 & 2 & 1 & 1 \end{bmatrix}$$

By reducing the matrix X to row-echelon form using elementary row operation or transformation, we obtain the equivalent matrix i.e.;

$$X \sim \begin{bmatrix} \boxed{1} & -1 & 1 & -1 & 1 \\ 0 & \boxed{1} & 1 & 1 & 0 \\ 0 & 0 & \boxed{2} & 7 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

From this echelon matrix, we conclude that the rank, $r = 3 < m$, where $m = 5$, i.e. the number of vectors, therefore the vectors;

$$x^{(1)}, x^{(2)}, x^{(3)}, x^{(4)}, x^{(5)}$$

are linearly dependent. Hence, the system $A\underline{X} = \underline{0}$ has a non-trivial solution since, $x^{(4)}$ and $x^{(5)}$ are free variables.

Remark 36 *Let X be the echelon form of an $m \times n$ matrix A, and let $r =$ rank of X. Then the rows of X are linearly independent and so are the column that carries the non-zero pivots. Since, a row rank is equal to a column rank, it therefore follows.*

Theorem 11.3.2. *The rows of an $n \times n$ identity matrix I_n are linearly independent. In general, any $V_n(F)$ is spanned by n unit vectors.*

Proof. Let $e_1 = (1, 0, ..., 0), e_2 = (0, 1, ..., 0), ..., e_n = (0, 0, ..., 1)$ be n-unit vectors. Indeed, any vector of $V_n(F)$ is a linear combination of these, because;

$$(\alpha_1, \alpha_2, ..., \alpha_n) = \alpha_1 e_1 + \alpha_2 e_2 + ... + \alpha_n e_n$$

where $\alpha_i (i = 1, 2, ...n)$ are scalars. But, $V_n(F)$ cannot be spanned by fewer than n vectors. This justifies calling $V_n(F)$ an $n-$*dimensional* vector space over the field F.

Not only do $e_1, e_2, ..., e_n$ generate the whole of $V_n(F)$; in addition,

$$\alpha_1 e_1 + \alpha_2 e_2 + ... + \alpha_n e_n = 0$$

if and only if,

$$(\alpha_1, \alpha_2, ..., \alpha_n) = (0, 0, ..., 0)$$

that is if $\alpha_1 = \alpha_2 = ... = \alpha_n = 0$ which completes the proof. □

Example 11.3.3

For what values of w are the vectors, $(w, 1, 0)$, $(0, w, 0)$ and $(0, 0, w+3)$ linearly independent?

Solution

If the vectors $(w, 1, 0)$, $(0, w, 0)$ and $(0, 0, w+3)$ are linearly independent, then;

$$\begin{vmatrix} w & 0 & 0 \\ 1 & w & 0 \\ 0 & 0 & w+3 \end{vmatrix} \neq 0$$

i.e.,

$$w(w^2 + 3w) \neq 0$$

Hence, if $w \neq 0$ or $w \neq -3$, the three vectors are linearly independent.

Example 11.3.4

Show that the vectors, $v_1 = (1, 1, 1)$, $v_2 = (0, 1, 1)$ and $v_3 = (0, 1, -1)$ are linearly independent, and that they generate $\mathbb{R}^3$.

Solution

We shall show that the relation;

$$\alpha v_1 + \beta v_2 + \gamma v_3 = 0 \qquad (i)$$

implies that $\alpha = \beta = \gamma = 0$
where α, β, and γ are scalars.

By substituting for v_1, v_2, v_3 in (i), we obtain;

$$\alpha(1, 1, 1) + \beta(0, 1, 1) + \gamma(0, 1, -1) = 0$$

i.e.,

$$\left.\begin{array}{rcl} \alpha & = & 0 \\ \alpha + \beta + \gamma & = & 0 \\ \alpha + \beta - \gamma & = & 0 \end{array}\right\} \qquad (ii)$$

which yield $\alpha = \beta = \gamma = 0$.

Thus, the vectors v_1, v_2 and v_3 are linearly independent.

Now, consider any other vector say, $\tilde{X} = (x_1, x_2, x_3) \in \mathbb{R}^3$. If $\tilde{X}$ depends on the given vectors v_1, v_2 and v_3, then, v_1, v_2 and v_3 generate $\mathbb{R}^3$. From,

$$(x_1, x_2, x_3) = \alpha(1,1,1) + \beta(0,1,1) + \gamma(0,1,-1),$$

we obtain,

$$\begin{aligned} x_1 &= \alpha \\ x_2 &= \alpha + \beta + \gamma \\ x_3 &= \alpha + \beta - \gamma \end{aligned}$$

i.e.,

$$\alpha = x_1,\ \beta = \frac{x_2 + x_3 - 2x_1}{2},\ \gamma = \frac{x_2 - x_3}{2}$$

Hence vectors, v_1, v_2 and v_3 spans $\mathbb{R}^3$.

Alternatively, you could solve as in Example 11.3.3 above by computing the determinant of the coordinates and hence the rank.

11.4 Bases

A *basis* of a vector space is a linearly independent subset which generates (spans) the whole space. A vector space is finitely dimensional if and only if it has finite bases. In other words, by a basis for a vector space v, we mean a set of vectors, v satisfying the following conditions:

(i) The set generates or spans v

(ii) The set is linearly independent

Furthermore, if $\varepsilon_1, \varepsilon_2, ..., \varepsilon_n$ form a basis of n-dimensional space and,

$$\tilde{X} = \alpha_1\varepsilon_1 + \alpha_2\varepsilon_2 + ... + \alpha_n\varepsilon_n$$

then the scalars, $\alpha_i = 1, (i, 2, ..., n)$ are called the *coordinates* of the vector $\tilde{X}$ in the given basis, $x_1, x_2, ..., x_n$. Note that the coordinates of the vector,

$$\tilde{X} = (x_1, x_2, ..., x_n)$$

are its coordinates in the basis of unit vectors;

$$e_j = (\delta_{1j}, \delta_{2j}, ..., \delta_{nj});\ (j = 1, 2, ..., n)$$

where

$$\delta_{nj} = \begin{cases} 1, & \text{if } n = j \\ 0, & \text{if } n \neq j \end{cases}$$

We thus have the basic expansion;

$$\tilde{X} = x_1 e_1 + x_2 e_2 + ... + x_n e_n$$

This basis of unit vectors in $\mathbb{R}^n$, $e_j (j = 1, 2, ..., n)$ will be called the *initial basis* of the space or more commonly, the *standard ordered basis* for $\mathbb{R}^n$

Example 11.4.1

(i) The domain, $F(x)$ of a polynomial form, in an indeterminate x over a field F is a linear space over F; since all the conditions for a vector space are satisfied in $F(x)$. The definition of equality applied to the equation $P(x) = 0$ implies that the powers, $1, x, x^2, x^3, ...$ are linearly independent over F. Hence, $F(x)$ has an infinite bases consisting of these powers, for any vector (polynomial form) can be expressed as a linear combination of a finite subset of these bases.

(ii) By Corollary 11.3.1, any vector space which is spanned or generated by a finite subset, linearly dependent or not, has finite bases; hence, we note in passing that a vector space is finitely-dimensional if and only if it can be generated by a finite subset.

Theorem 11.4.1. *Any vector in a space V can be expressed uniquely as a linear combination of the vectors of a given basis.*

Proof. Since the bases, $\varepsilon_1, \varepsilon_2, ..., \varepsilon_n$ spans the space, every vector, $\tilde{X}$ is a linear combination;

$$\tilde{X} = x_1 \varepsilon_1 + x_2 \varepsilon_2 + ... + x_n \varepsilon_n$$

If $\tilde{X}$ is also equal to a second such combination, say,

$$\tilde{X} = y_1\varepsilon_1 + y_2\varepsilon_2 + ... + y_n\varepsilon_n$$

then, by subtraction and recombination we obtain;

$$(x_1 - y_1)\varepsilon_1 + (x_2 - y_2)\varepsilon_2 + ... + (x_n - y_n)\varepsilon_n = \tilde{X} - \tilde{X} = 0$$

Since the basis vectors, $\varepsilon_i (i = 1, 2, ..., n)$ are independent, this implies that each coefficient $x_i - y_i = 0$, so that each $x_i = y_i$, which shows that the expression of $\tilde{X}$ as a linear combination of the ε_i is unique. □

Theorem 11.4.2. *Any independent set of elements of a finite-dimensional vector space V is part of a basis.*

Proof. Let the independent set be $v_1, v_2, ..., v_m$ and let $\varepsilon_1, \varepsilon_2, ..., \varepsilon_n$ be a basis for V. From the set, $[v_1, v_2, ..., v_m;\ \varepsilon_1, \varepsilon_2, ..., \varepsilon_n]$, we can extract an independent subset which also spans V (is a basis for V), by deleting one-by-one terms which are linear combinations of their predecessors. Since the V_i are independent, no V_i will be deleted, and so the resulting bases will include every $V_i (i = 1, 2, ...m)$. □

Example 11.4.2

Show that in the vector space $\mathbb{R}^n$, the following set of vectors:
$e_1 = (1, 0, 0, ...0)$, $\quad e_2 = (0, 1, 0, ..., 0)...e_n = (0, 0, ..., 1)$ is a basis for $\mathbb{R}^n$.

Solution

Suppose $\alpha_1 e_1 + \alpha_2 e_2 + ... + \alpha_n e_n = 0$, for every $\alpha_i \in \mathbb{R};\ i = 1, 2, ..., n$, then,

$$(\alpha_1, \alpha_2, ..., \alpha_n) = (0, 0, ..., 0)$$

and by the definition of equality of vectors, we have that;

$$\alpha_1 = \alpha_2 = ... = \alpha_n = 0$$

This implies that $[e_1, e_2, ..., e_n]$ is a linearly independent subset of $\mathbb{R}^n$. To show that $[e_1, e_2, ..., e_n]$ generates or spans $\mathbb{R}^n$, let

$\tilde{X} = (x_1, x_2, ..., x_n)$ be any vector in R^n. Then, it can easily be shown that;

$$\tilde{X} = x_1e_1 + x_2e_2 + ... + x_ne_n$$

i.e. any vector in $\mathbb{R}^n$ can be written as a linear combination of the vectors, $e_1, e_2, ..., e_n$. Hence $[e_1, e_2, ..., e_n]$ spans $\mathbb{R}^n$. Therefore, $[e_1, e_2, ..., e_n]$ is linearly independent subset of $\mathbb{R}^n$ which spans $\mathbb{R}^n$ and so is a basis for $\mathbb{R}^n$. In particular, this basis is called the *usual or standard ordered basis* for $\mathbb{R}^n$.

Example 11.4.3

Let $V = \left\{ \begin{bmatrix} p & q \\ r & s \end{bmatrix} : p, q, r, s \in \mathbb{R} \right\}$ be a vector space over $\mathbb{R}$. Find the basis for V.

Solution

The set,

$$S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

spans the vector space v and are linearly independent subset of $\mathbb{R}^2$. Thus, S form a basis for V, hence;

$$v = \begin{bmatrix} p & q \\ r & s \end{bmatrix} = p \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + q \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + r \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + s \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Example 11.4.4

Show that the vectors $\tilde{V_1} = (1, 2, 3)$; $\tilde{V_2} = (0, 1, 2)$ and $\tilde{V_3} = (0, 0, 1)$ form a basis for the vector space $V_3(F)$.

Solution

To show that the set are linearly independent, let α, β and γ be scalars such that,

$$\alpha V_1 + \beta V_2 + \gamma V_3 = 0 \qquad (i)$$

which implies that,

$$\alpha = \beta = \gamma = 0 \qquad (ii)$$

By substituting for V_1, V_2, V_3 in (i), we obtain

$$\alpha(1,2,3) + \beta(0,1,2) + \gamma(0,0,1) = 0$$

i.e.

$$\left.\begin{array}{rcl} \alpha & = & 0 \\ 2\alpha + \beta & = & 0 \\ 3\alpha + 2\beta + \gamma & = & 0 \end{array}\right\} \qquad (iii)$$

which yields $\alpha = \beta = \gamma = 0$
Thus, the vectors $\tilde{V}_1$, $\tilde{V}_2$ and $\tilde{V}_3$ are linearly independent.

Now, consider any other vector, say $\tilde{X} = (x_1, x_2, x_3)$ in $\mathbb{R}^3$. If $\tilde{X}$ depends on the given vectors $\tilde{V}_1$, $\tilde{V}_2$ and $\tilde{V}_3$, then $\tilde{V}_1$, $\tilde{V}_2$ and $\tilde{V}_3$ generate $\mathbb{R}^3$.
Thus, from

$$(x_1, x_2, x_3) = \alpha(1,2,3) + \beta(0,1,2) + \gamma(0,0,1)$$

we obtain

$$\begin{array}{rcl} x_1 & = & \alpha \\ x_2 & = & 2\alpha + \beta \\ x_3 & = & 3\alpha + 2\beta + \gamma \end{array}$$

$$\therefore \qquad \alpha = x_1,\ \beta = x_2 - 2x_1,\ \gamma = x_3 - 2x_2 + x_1$$

Hence, the vectors $\tilde{V}_1$, $\tilde{V}_2$ and $\tilde{V}_3$ spans $\mathbb{R}^3$.

Alternatively, to show that the set are linearly independent, we need to show that the set of the coordinates is non-singular, i.e. determinant is not equal to zero, and hence compute the rank such that the vectors $\tilde{V}_1$, $\tilde{V}_2$ and $\tilde{V}_3$ spans $\mathbb{R}^k$; where $k =$ rank

11.5 Dimension

Let V be a vector space over the field F, then the *dimension* of the finitely-generated vector space V denoted by $d[V]$ is the number of elements or vectors in any bases for V. In other words, if the number of elements in any bases for V is n, we then say that V is an n-dimensional vector space. Thus $\mathbb{Q}^n$, $\mathbb{R}^n$ and $\mathbb{C}^n$ are n-dimensional vector spaces. This is why we sometimes denote an n-dimensional vector space V over a field F by $V_n(F)$. The preceding paragraphs suggest the idea that every bases of $V_n(F)$ has n elements.

Theorem 11.5.1. *Any two bases for a finitely generated vector space V contain the same number of elements or vectors. This number is called the dimension of the vector space. This theorem can as well be put in the following way: Suppose that $S = \{V_1, V_2, ..., V_n\}$ and $T = \{U_1, U_2, ..., U_r\}$ are bases for the same vector space V. Then $n = r$.*

Proof. Suppose $n < r$, since $V_1, V_2, ..., V_n$ is a bases for V, and for each $k = 1, 2, ..., r, U_k \in V$, we have that each U_k is a linear combination of $V_{i's}$. Similarly, each V_i is a linear combination of $U_{k's}$. Consider:

$$\begin{aligned} \sum_{i=1}^{r} a_i V_i &= \sum_{i=1}^{r} a_i \left[\sum_{k=1}^{n} \beta_{ik} V_k \right]; \quad a_i, \beta_{ik} \in \mathbb{R} \\ &= \sum_{k=1}^{n} \left[\sum_{i=1}^{r} a_i \beta_{ik} \right] V_k \end{aligned}$$

where

$$\sum_{i=1}^{r} a_i V_i = 0$$

Thus,

$$\sum_{k=1}^{n} \left[\sum_{i=1}^{r} a_i \beta_{ik} \right] V_k = 0$$

It follows that the $V_{i's}$ form a basis for V, so this implies that, for every $k = 1, 2, ..., n$;

$$\sum_{i=1}^{r} a_i \beta_{ik} = 0$$

This is equivalent to the following system of equations:

$$\begin{array}{ccccccccc} a_1\beta_{11} & + & a_2\beta_{21} & + & ... & + & a_r\beta r1 & = & 0 \\ a_1\beta_{12} & + & a_2\beta_{22} & + & ... & + & a_r\beta_{r2} & = & 0 \\ \vdots & & ... & ... & ... & ... & ... & ... & ... \\ a_1\beta_{1n} & + & a_2\beta_{2n} & + & ... & + & a_r\beta_{rn} & = & 0 \end{array}$$

which is a homogeneous system of n equations in r unknowns, but $n < r$.

So, the system has a non-trivial solution i.e. there exists $a_1, a_2, ...,$ a_r not all zero, such that the $RHS = 0$. But this contradicts the fact that the $U_{k's}$ form a basis for V and therefore need to be linearly independent. In particular, if V has two bases (sets spanning V and independent) consisting of n and r elements respectively, both $n \geq r$ and $r \geq n$ hold. Hence $n = r$. □

Corollary 11.5.1. *In an n-dimensional vector space over a field $F, V_n(F)$, any $n+1$ vectors are linearly dependent. In other words, no set of more than n vectors can be linearly independent, and no set less than n vectors can span the vector space.*

So, we have that a basis is a maximal linearly independent set and a minimal spanning set. Thus, Corollary 11.5.1 gives us a procedure for constructing a basis for any vector space.

Theorem 11.5.2. *If S and T are two subspaces of a vector space V, then their dimensions satisfy;*

$$d[S] + d[T] = d[S \cap T] + d[S + T]$$

Proof. Let $\alpha_1, \alpha_2, ..., \alpha_n$ be a basis for $S \cap T$; by theorem 11.4.2, S and T have bases, $\alpha_1, \alpha_2, ..., \alpha_n$; $\beta_1, \beta_2, ..., \beta_r$ and $\alpha_1, \alpha_2, ..., \alpha_n$; $\gamma_1, \gamma_2, ...\gamma_p$ respectively. Clearly, the α_i, β_j, and γ_k together span $S + T$. They are even a basis since,

$$a_1\alpha_1 + a_2\alpha_2, ..., +a_n\alpha_n + b_1\beta_1 + ... + b_r\beta_r + ... + c_p\gamma_p = 0$$

implies that

$$\sum_{j=1}^{r} b_j\beta_j = -\sum_{i=1}^{n} a_i\alpha_i - \sum_{k=1}^{p} c_k\gamma_k \text{ is in } T$$

where $\sum b_i\beta_i$ is in $S \cap T$ and so,

$$\sum b_i\beta_i = \sum d_i\alpha_i; \text{ for some } d_i \in \mathbb{R}$$

Hence (then α_i and β_j being independent) every $b_j = 0$.
Similarly, every $c_k = 0$;
Now, by substitution,

$$\sum a_i\alpha_i = 0, \text{ and every } a_i = 0.$$

This shows that the α_i, β_j, and γ_k are a basis for $S + T$. □

Example 11.5.1

Find the dimension and a basis for the null space of the matrix:

$$A = \begin{bmatrix} -2 & 4 & 6 \\ 1 & 0 & -1 \\ 2 & -4 & -6 \end{bmatrix}$$

Solution

The null space of A is the solution space of $A\underline{X} = \underline{0}$ and reducing to echelon form, we have,

$$A \sim \begin{bmatrix} -2 & 4 & 6 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, $A\underline{X} = \underline{0}$ is equivalent to;

$$\begin{bmatrix} -2 & 4 & 6 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Hence, X_3 is a free variable, so that if we let $X_3 = \alpha$, then

$$2X_2 = -2\alpha \text{ or } X_2 = -\alpha \text{ and } X_1 = \alpha$$

$\therefore$ Null space of $A = \{\alpha(1, -1, 1) : \alpha \in \mathbb{R}\}$.
Thus, a basis for the null space of A is $B = (1, -1, 1)$ and dimension of the null space of A is $d(A) = 2$.

Remark 37 *The row space of a matrix $A = [a_{ij}]_{m,n}$ is the space generated by the rows of A and it is a subspace of $\mathbb{R}^n$. While the column space of a matrix, $A = [a_{ij}]_{m,n}$ is the space generated by the column of A and it is a subspace of $\mathbb{R}^m$.*

Consequently, if $m = n$, then the row space and the column space of A are both subspaces of $\mathbb{R}^n$.

Example 11.2.10

Find the bases for the row space and the column space of the matrix:
$A = \begin{bmatrix} -1 & 3 & 2 & -4 \\ 0 & 1 & 3 & -5 \\ 2 & -6 & -4 & 8 \end{bmatrix}$

Solution

The echelon form of A,. i.e.

$$A \sim \begin{bmatrix} -1 & 3 & 2 & -4 \\ 0 & 1 & 3 & -5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

i.e. Row Space of A has a basis $\{R_1, R_2\}$ and column space of A is generated by the columns 1, 2, 3 and 4.

Thus, columns 1 and 2 form a basis for the column space of A.

Remark 38 *For any echelon matrix A, the rows with the non-zero pivots form a basis for the row space of A, while the columns with the non-zero pivots form a basis for the column space of A.*

Example 11.5.3

Find the dimension and a basis for the subspace of $\mathbb{R}^4$ generated by the vectors
$(1, 0, 2, -1), (3, -1, -2, 0), (1, -1 - 6, 2), (0, 1, 8, -3)$.

Solution

Let the matrix of the vectors be A such that,

$$A = \begin{bmatrix} 1 & 3 & 1 & 0 \\ 0 & -1 & -1 & 1 \\ 2 & -2 & -6 & 8 \\ -1 & 0 & 2 & -3 \end{bmatrix}$$

Then, the echelon form of A, i.e.

$$A \sim \begin{bmatrix} 1 & 3 & 1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ the bases $= \{(1, 3, 1, 0), (0, 1, 1, -1)\}$ and dimension $= 2$.

Chapter 12

Vectors

Trigonometry forms the bedrock of vectors - the subject of this chapter. In Chapter 8, we discussed trigonometry as the study of the relations of the sides and angles of triangles and of the methods of applying these relations in the solution of problems involving triangles. It is the combinations of the knowledge of trigonometry together with magnitude of physical quantities that gives rise to vector quantities. As in trigonometry, vectors find its applications in the design of navigational systems and surveying. Also, it is widely used in weather forecasting to find the direction and magnitude of wind-storm, cyclone, hurricane etc.

12.1 Scalar and Vector Quantities

A *scalar quantity*, or simply a scalar, is a physical quantity or pure number which has magnitude only, it is not related to any definite direction in space. Examples are mass, time, real number, volume, work, temperature and speed. To specify a scalar, we need a unit quantity of the same type and the ratio (m) which the given quantity bears to this unit. The number m is called the measure of the quantities in terms of the chosen unit.

On the other hand, a *vector quantity*, or a *vector*, is a line representing a physical quantity which has magnitude and is related to a definite direction in space. Examples are displacement, force, momentum, velocity and acceleration. To specify a vector, we need not only a unit quantity of the same kind considered apart from direction, and a number which is the measure of the original quantity in terms of this unit, but also a statement of its direction.

We may distinguish between three sets of vectors by their effects:

(i) Free Vectors: They have, magnitude and direction, but have no particular position associated with them. An example of a free vector is displacement; for instance, a displacement vector '10 meters South' is the same in Nigeria as in Ghana.

(ii) Line Located Vectors: A *line located vector* is one which is located along a straight line. For instance, if force acts on a rigid body it is clear that the force can only be moved along its line of action without changing its effect on the body. In other words, *force* is an example of line located vectors.

(iii) Point Located Vectors: The vector of a unit magnitude in the direction of steepest slope at a point on a hill applies only to that point. Such a vector is an example of a point located vector.

12.1.1 Representation of a Vector

In vector analysis, we require elements involving both number and direction-directed numbers; for this purpose, since a vector is characterized completely by magnitude and direction, it can be conveniently represented by a segment of a straight line. The straight line OA, or more precisely $\vec{OA}$ represents a vector, whose magnitude is represented by the length OA, and is indicated by the direction O to A; the arrow in $\vec{OA}$ gives precisely, the direction. To distinguish a vector quantity from a scalar, it is the convention to represent the former, by letters printed in 'Clarendon' type, i.e. in bold type (upper or lower case). Pictorially, the velocity of a truck moving along a road at 60 km/hr to the East is represented in Figure 12.1 below.

Observe that the vector (i.e. velocity) is completely represented by a line OA in this direction, while the arrow drawn on OA is used to indicate the direction of motion of the truck.

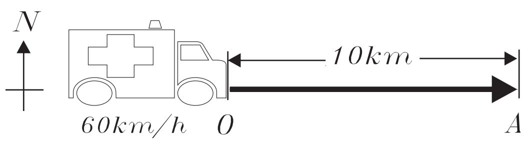

Figure 12.1: A truck moving eastward along a road at 60km/h

The vector $\vec{OA}$, can be represented or written in several ways as **OA**, $\underline{\mathbf{A}}$ or **A**, $\underline{\mathbf{a}}$ or **a**. In representing several vectors of the same type in a particular system, the same relationship between vector magnitude and line length must be observed or the system will be inconsistent.

12.1.2 Types of Vectors

(i) Position Vectors: Let O be an arbitrary fixed point taken as the origin, if A be any other point then $\vec{OA}$ is said to be the *position vector* with respect to O. Note that it must not be confused with the point A itself; nor with the directed line segment $\vec{AO}$, which is not a position vector since it starts at A rather than at O. If $\underline{a}$, $\underline{b}$, $\underline{c}$, are position vectors of A, B, C, with reference to O as origin, then,

$$\vec{OA} = \underline{a}; \quad \vec{OB} = \underline{b}; \quad \vec{OC} = \underline{c}$$

However, if position vectors of several points are given without reference to any origin, they are supposed to be referred to the same origin. If $\vec{OA}, \vec{OB}, \vec{OC}$ are three vectors directed away from the same point O, they are called *co-initial*.

(ii) Null Vectors: The position vector $\vec{OO}$ is special - it has no length, and has arbitrary direction. We call it the null or zero vector and is denoted the by Clarendon symbol for zero, O (more correctly, we should call it the zero position vector). However, vectors other than the zero vector are called *proper vectors*

(iii) Unit Vector: The *Modulus* or *Magnitude*, of a vector is the positive number which is the measure of its length. The modulus of the vector $\underline{a}$ is denoted by $|\underline{a}|$, and by the corresponding symbol in italics. Therefore, a *unit* vector is any vector whose magnitude is unity (one). Unit vectors are used to specify directions; they are extensively used when vectors are specified as a sum of components in particular directions. There are several ways of writing unit vectors, and among these are: $\underline{\hat{u}}$ and $\underline{e}_u$, both of which represent the unit vector in the direction of $\underline{u}$. However for consistency, we shall restrict ourselves to the use of $\underline{e}_u$ in our discussion in this text.

12.1.3 Algebraic Operations of Vectors

(a) Equality of Vectors

Two vectors $\underline{a}$, $\underline{b}$ are said to be *equal* if they have equal magnitude or lengths and in the same direction; this is denoted by

$$\underline{a} = \underline{b}$$

The vector which has the same modulus as $\underline{a}$, but the opposite direction, is called the *negative* of $\underline{a}$, and is denoted by

$$-\underline{a}; \quad \text{negative of } \underline{a}$$

(b) Parallel Vectors

Let the vectors $\underline{a}, \underline{b}$ be represented by $\vec{OA}$ and $\vec{OB}$ with angle of inclination θ^o as follows:

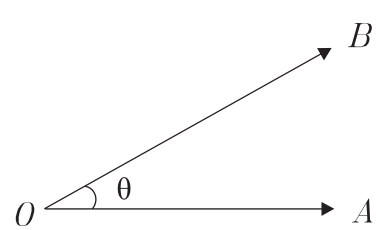

Figure 12.2: Inclination of vectors

Then the inclination of the vectors, or the angle between them θ, is defined as that angle AOB which does not exceed π. Thus

θ denotes this inclination, where

$$0 \leq \theta \leq \pi$$

When the inclination, $\theta = \frac{\pi}{2}$, the vectors are said to be *perpendicular*; when it is 0 or π they are *parallel*.

The term *collinear* will be used as synonymous with parallel; in that vectors which are parallel to the same line are said to be collinear, and the line is said to be the *line of vectors*. Further, three or more vectors are said to be *coplanar* when they are parallel to the same plane; and any plane parallel to this, is called *the plane of the vectors*. Vectors as defined above are examples of free vectors since the values of such vectors depend only on its magnitude and direction, and is independent of its position in space.

(c) Addition and Subtraction of Vectors

(i) Vector Addition: The addition of vectors is defined by the triangle law:

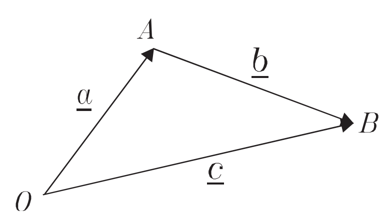

Figure 12.3: Sum or resultant of vectors

From the Figure 12.3 above, O, A, B are three points such that $\vec{OA} = \underline{a}$, $\vec{AB} = \underline{b}$. Then $\vec{OB}$ is said to be the *sum* or *resultant* of the vectors $\underline{a}$ and $\underline{b}$ or by definition,

$$\vec{OB} = \vec{OA} + \vec{AB} \tag{12.1.1}$$

or

$$\underline{c} = \underline{a} + \underline{b}; \quad \text{where } \vec{OB} = \underline{c}$$

The sign of equality, +, borrowed from algebra, has a special meaning, indicating what may be called compounding of vectors. It does not mean the arithmetical sum, except when O, A, C are collinear.

(ii) Subtraction of Vectors: By definition, we already know that the vectors $\underline{b}$ and $-\underline{b}$ have the same modulus, but have opposite directions. Thus, the subtraction of $\underline{b}$ from $\underline{a}$ is to be understood as the addition of $-\underline{b}$ to $\underline{a}$. We denote this by

$$\underline{a} - \underline{b} = \underline{a} + (-\underline{b}) \tag{12.1.2}$$

Thus using the triangle law of addition we have;

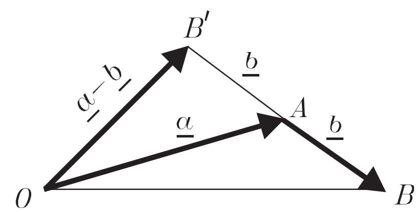

Figure 12.4: Triangle law of addition of vectors

From Figure 12.4, if A is the mid-point of BB',

$$\begin{aligned} \vec{AB}' &= -\vec{AB} \\ \vec{OB}' &= \vec{OA} + \vec{AB}' \\ \vec{OB}' &= \vec{OA} - \vec{AB} \end{aligned}$$

Thus, to subtract the vector $\underline{b}$ from $\underline{a}$, the direction of b is reversed and added. In particular, if $b = a$, we have;

$$\underline{a} - \underline{a} = \vec{OA} + \vec{AO} = \vec{OO} \tag{12.1.3}$$

Obverse that $\vec{OO}$ is a degenerate vector of zero length which is a null vector. Its direction is indeterminate and all null vectors are considered equal.

♦ *Parallelogram Law for Addition and Subtraction of Vectors:*

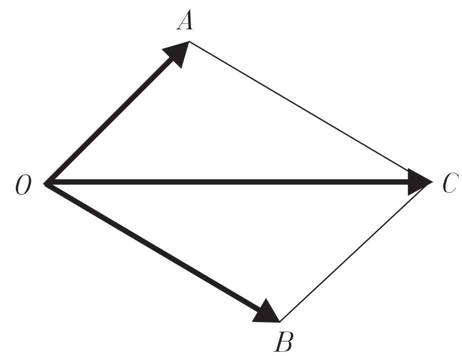

Figure 12.5: Parallelogram law

The parallelogram law for vector addition states that the sum of the vectors $\vec{OA}$ and $\vec{OB}$ is given by $\vec{OC}$, where $OACB$ is a parallelogram, as in Figure 12.5.
So that,

$$\vec{OA} + \vec{OB} = \vec{OC} \tag{12.1.4}$$

This law gives the same sum as the triangle law, as we now show below:
By the triangle law $\vec{OA} + \vec{AC} = \vec{OC}$ and from the definition of equal vectors;

$$\begin{aligned} \vec{AC} &= \vec{OB} \\ \text{i.e.,} \quad \vec{OA} + \vec{OB} &= \vec{OC} \end{aligned}$$

which proves that the two laws are equivalent.

♦ *Addition of Several Vectors:* Suppose we wish to add $\vec{OA}, -\vec{BA}, \vec{BA}, \vec{BC}, \vec{CD}$, and $\vec{DE}$. We can picture the sum as in Figure 12.6.
From Figure 12.6 below,

$$\begin{aligned} -\vec{BA} &= \vec{AB} \\ \text{and} \qquad \vec{OA} + \vec{AB} &= \vec{OB} \\ \vec{OB} + \vec{BC} &= \vec{OC}, \text{ and so on.} \end{aligned}$$

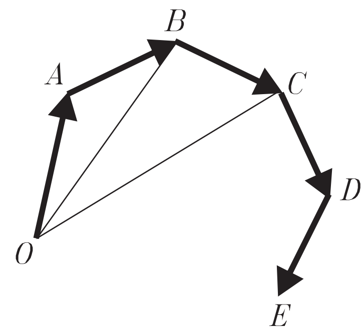

Figure 12.6: Addition of several vectors

Thus,

$$\vec{OA} + \vec{AB} + \vec{BC} + \vec{CD} + \vec{DE} = \vec{OE} \qquad (12.1.5)$$

Similarly, we can perform additions of this type without a diagram, by grouping the vectors appropriately; provided the vectors can be strung together head to tail. The vector sum, called the resultant, is the vector represented by going from the first to the last point of the string. The vectors $\vec{OA}, \vec{AB}, \vec{BC}, \vec{CD}$, and $\vec{DE}$ are known as *component vectors* of the resultant $\vec{OE}$.

Furthermore, if we add the vector $\vec{OE}$ to the above list, the resultant vector would be the null or zero vector. In other words, any closed loop of vectors (head to tail) is equal to the null vector. It follows that whenever several vectors are to be added (or subtracted), it can be shown that the order of addition (or subtraction) has no effect on the result.

Example 12.1.1

Find the sum of the vectors $3\vec{OA},\ 4\vec{BC},\ \vec{AO},\ 2\vec{AB},\ 2\vec{OB}$.

Solution

$$\begin{aligned}
3\vec{OA} + 4\vec{BC} &+ \vec{AO} + 2\vec{AB} + 2\vec{OB} \\
&= \vec{OA}(3-1) + 4\vec{BC} + 2\vec{AB} + 2\vec{OB} \\
&= 2\vec{OA} + 2\vec{AB} + 2\vec{OB} + 4\vec{BC} \\
&= 2(\vec{OA} + \vec{AB}) + 2\vec{OB} + 4\vec{BC} \\
&= 2\vec{OB} + 2\vec{OB} + 4\vec{BC} \\
&= 4\vec{OB} + 4\vec{BC} \\
&= 4(\vec{OB} + \vec{BC}) \\
&= 4\vec{OC}
\end{aligned}$$

Example 12.1.2

If $\vec{OB}$ bisects $\vec{AC}$, show that $\vec{OA} - \vec{OB} - \vec{BC} = \underline{O}$, when O is any point.

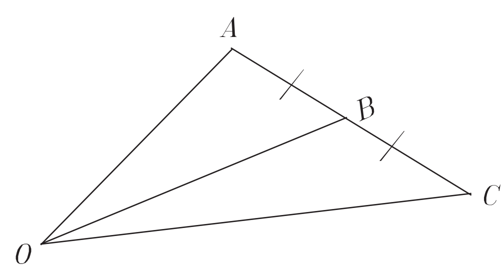

Figure 12.7:

Solution

Since $\vec{BA}$ and $\vec{BC}$ are equal in magnitude, then it is either $\vec{BA} = -\vec{BC}$ or $\vec{AB} = \vec{BC}$.
Now suppose, $\vec{BA} = -\vec{BC}$, then;

$$\begin{aligned}
\vec{OA} + \vec{AB} &= \vec{OB} \\
\text{i.e} \quad \vec{OA} - \vec{BC} &= \vec{OB} \ \textit{(triangle law)} \\
\therefore \quad \vec{OA} - \vec{BC} - \vec{OB} &= \underline{O}
\end{aligned}$$

(d) Multiplication of Vector by Scalar

From the law of addition of vectors, it follows that $\underline{a}+\underline{a}+\underline{a}+\ldots$ to n terms is a vector in the same direction as $\underline{a}$. It is denoted by $n\underline{a}$. We generalize this by the following definition:

The product of a vector $\underline{a}$ and a scalar α denoted by $\alpha\underline{a}$ or $\underline{a}\alpha$, is defined as a vector whose magnitude is $|\alpha|$ times that of $\underline{a}$, and whose direction is the same as that of $\underline{a}$, i.e,

$$|\alpha\underline{a}| = |\alpha||\underline{a}| \tag{12.1.6}$$

where

$$|\alpha| = \begin{cases} \alpha \text{ for } \alpha > 0, \text{ (in which case } \underline{a} \text{ and } \alpha\underline{a} \\ \text{have the same direction)} \\ 0 \text{ for } \alpha = 0, \text{ (in which case } \alpha\underline{a} = \underline{0}) \\ -\alpha \text{ for } \alpha < 0, \text{ (in which case } \underline{a} \text{ and } \alpha\underline{a} \\ \text{ have opposite direction)} \end{cases}$$

From the above definition it follows that

$$\alpha(-\underline{a}) = (-\alpha)\underline{a} = -\alpha\underline{a}, \quad (-\alpha)(-\underline{a}) = \alpha\underline{a} \tag{12.1.7}$$

Thus, if $\underline{e}_u$ is the unit vector with the same direction as $\underline{a}$, and $\overline{a}$ is $|\underline{a}|$, then

$$\underline{a} = \overline{a}\underline{e}_u, \quad \alpha\underline{a} = \alpha(\overline{a}\underline{e}_u) = (\alpha\overline{a})\underline{e}_u \tag{12.1.8}$$

(e) Division of Vector by Scalar

Division of a vector by a scalar, α is defined as multiplication of the vector by $\frac{1}{\alpha}$. Thus if $\underline{e}_u$ is the unit vector in the direction of $\underline{a}$, then;

$$\underline{e}_u = \frac{\underline{a}}{\underline{a}} \tag{12.1.9}$$

and if $\underline{b}$ is parallel to $\underline{a}$, then;

$$\underline{b} = \pm\underline{b}\frac{\underline{a}}{\underline{a}}; \tag{12.1.10}$$

according as the two vectors have the same direction or opposite directions.

12.1.4 Laws of Vector Algebra

Let $\underline{a}$, $\underline{b}$ and $\underline{c}$ be vectors and let $\alpha, \beta \in \mathbb{R}$, be scalars, then from the principles of vector addition subtractions and multiplication of a vector by scalar, the following rules are satisfied:

(i) $\underline{a} + \underline{b} = \underline{b} + \underline{a}$ *(Commutative law)*

(ii) $\underline{a} + (\underline{b} + \underline{c}) = (\underline{a} + \underline{b}) + \underline{c}$ *(Associative law with respect to addition)*

(iii) $\underline{a} + 0 = \underline{a}$ *(Additive Identity)*

(iv) $\underline{a} + (-\underline{a}) = 0$ *(Additive Inverse)*

(v) $\alpha\underline{a} = \underline{a}\alpha$ *(Commutative law with respect to multiplication)*

(vi) $\alpha(\beta\underline{a}) = (\alpha\beta)\underline{a}$ *(Associative law with respect to multiplication)*

(vii) $(\alpha + \beta)\underline{a} = \alpha\underline{a} + \beta\underline{a}$ *(Right Distributive law)*

(viii) $\alpha(\underline{a} + \underline{b}) = \alpha\underline{a} + \alpha\underline{b}$ *(Left Distributive law)*

(ix) $\underline{a} \cdot \underline{e} = \underline{a}$ *(Multiplicative Identity)*

(x) $\underline{a} \cdot \underline{a}^{-1} = \underline{e}, \quad \underline{a} \neq 0$ *(Multiplicative Inverse)*

These rules correspond exactly with the rules of ordinary algebra.

12.2 Component of Vectors

12.2.1 Vectors in Two Dimensions

In vector analysis, it is often useful to resolve all the vectors in particular directions, called reference directions, and to express each vector as a sum of *component vectors* in these directions.

Let OA, OB, OR be three coplanar lines; let $\vec{OA} = \underline{a}$, $\vec{OB} = \underline{b}$, $\vec{OR} = \underline{r}$; $\underline{a}$ and $\underline{b}$ being non-collinear vectors.

Complete the parallelogram $OA'RB'$ with OR as diagonal and adjacent sides along OA and OB parallel to $\underline{a}$ and $\underline{b}$ respectively.

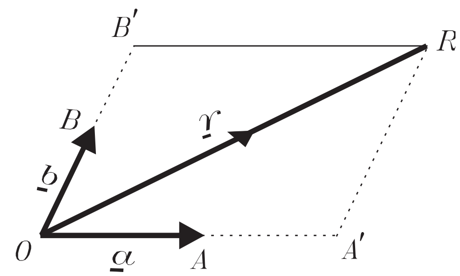

Figure 12.8: Vectors in two dimension

Clearly

$$\vec{OA'} = x.\vec{OA} = x\underline{a}, \quad \text{and} \quad \vec{OB'} = y\underline{b}$$

where x and y are scalars.

Also,

$$\vec{OR} = \vec{OA'} + \vec{A'R} = \vec{OA'} + \vec{OB'}$$

or

$$\underline{r} = x\underline{a} + y\underline{b} \tag{12.2.1}$$

The vectors $x\underline{a}$ and $y\underline{b}$ are said to be the components of $\underline{r}$ along the directions of $\underline{a}$ and $\underline{b}$. In particular, if

$$x = y = 1, \quad \text{then,} \quad \underline{r} = \underline{a} + \underline{b} \tag{12.2.2}$$

or $\underline{r}$ is the resultant or sum of $\underline{a}$ and $\underline{b}$.

If the directions of $\underline{a}$ and $\underline{b}$ are at right angles, then the components of $\underline{r}$, viz: $x\underline{a}$ and $y\underline{b}$ are also called the resolved parts of $\underline{r}$. The above relation is the expression of any vector $\underline{r}$ as a linear function of two coplanar, non-collinear vectors. This expression is unique.

12.2.2 Vectors in Three Dimension

We can extend our argument to three dimensions where we need three reference directions to specify any vector. In other words, any vector $\underline{r}$ can be expressed as the sum of three vectors, parallel to any three non-coplanar vectors. Let $\underline{a}$, $\underline{b}$, $\underline{c}$ be unit vectors in the three given non-coplanar directions. With any point O as the origin, take

$\vec{OR} = \underline{r}$, and on OR as diagonal, construct a parallelepiped with sides OA, OB, OC parallel to $\underline{a}, \underline{b}, \underline{c}$ respectively.

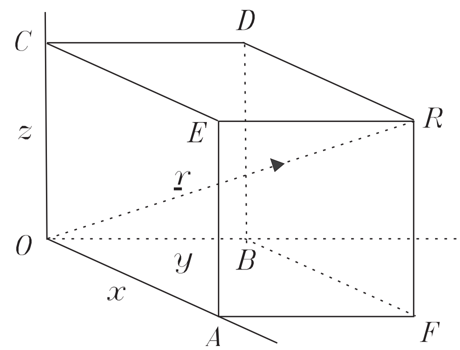

Figure 12.9: A parallelepiped

Now, let x, y, z be scalars such that $\vec{OA} = x\underline{a}$, $\vec{OB} = y\underline{b}$, $\vec{OC} = z\underline{c}$. The scalar x is positive or negative according as $\vec{OA}$ has the same direction as $\underline{a}$ or the opposite direction; and similarly for y and z. Thus expressing the given vector $\underline{r}$ as the sum of the three non-coplanar vectors we have;

$$\begin{aligned} \underline{r} &= \vec{OA} + \vec{AF} + \vec{FR} \\ &= \vec{OA} + \vec{OB} + \vec{OC} \\ &= x\underline{a} + y\underline{b} + z\underline{c} \end{aligned} \tag{12.2.3}$$

Then $\underline{r}$ is called the resultant or sum of the three vectors $x\underline{a}, y\underline{b}, z\underline{c}$, which are the *component vectors* of $\underline{r}$. The resolution of $\underline{r}$ is unique, because only one parallelepiped can be constructed on OR as diagonal with sides parallel to the given directions. Hence, if two vectors are equal, their corresponding components are equal. Conversely, if the correspondents components of two vectors are equal, the vectors correspondingly equal.

We note that, if $\underline{r}$ is regarded as the position vector of the point R relative to O, then x, y, z are the *Cartesian co-ordinate* of R with respect to the axes through 0 in the given directions.

12.3 Rectangular Resolution of Vectors in Space

The most important case of resolution of vectors is that in which the three directions are mutually perpendicular.

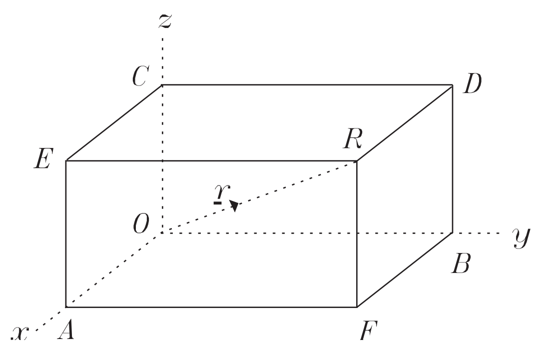

Figure 12.10: Rectangular vector in space

Let OX, OY, OZ be three straight lines, each perpendicular to the other two, i.e. to the plane of the other two, forming what is known as a right handed system of directions. Observe that OY and OZ are in the plane of the paper, and OX perpendicular to it pointing toward the reader. The unit vectors parallel to axes OX, OY, OZ are denoted by i, j, k; and the parallelepiped on OR as diagonal with sides OA, OB, OC parallel to i, j, k, is now rectangular.

Now, let (x, y, z) be the Cartesian co-ordinate of R; then

$$OA = x, \quad OB = y, \quad OC = z$$

So that we may write;

$$\vec{OA} = x\underline{i}, \quad \vec{OB} = y\underline{j}, \quad \vec{OC} = z\underline{k}$$

The number x being positive or negative according as $\vec{OA}$ has the same direction as i, or the opposite direction; and similarly for y and z. We can now express the given vector as;

$$\begin{aligned} \vec{OR} &= \vec{OA} + \vec{OB} + \vec{OC} \\ \text{or} \quad \underline{r} &= x\underline{i} + y\underline{j} + z\underline{k} \end{aligned} \tag{12.3.1}$$

The vectors $x_{\underline{i}}, y_{\underline{j}}, z_{\underline{k}}$ are called the *component vectors* of $\underline{r}$ along the mutually perpendicular directions OX, OY and OZ respectively. We observe that the vectors $x\underline{i}, y\underline{j}, z\underline{k}$ respectively are the *orthogonal* projections of $\underline{r}$ along these directions and $\underline{r}$ is the sum of these components. In rectangular co-ordinates, x, y, z are called the *rectangular components* or *resolute* or *resolved parts* of $\underline{r}$ along OX, OY, OZ respectively.

If $\underline{r}$ makes angles α, β, γ respectively with the axes, then $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are called the *direction cosines* of $\underline{r}$ and are often denoted by l, m, n. Clearly,

$$x = \underline{r}\cos\alpha = \ell\underline{r};\; y = \underline{r}\cos\beta = m\underline{r},\; z = \underline{r}\cos\gamma = n\underline{r} \qquad (12.3.2)$$

So that the resolute of a vector in a direction, inclined at an angle $\theta$to it, is obtained by multiplying the modulus of the vector by $\cos\theta$.

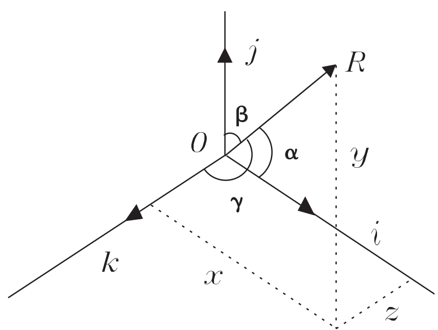

Figure 12.11: Component vectors

The unit vector in the direction of $\underline{r}$ is:

$$\begin{aligned} \frac{1}{r}\underline{r} &= \frac{xi}{r} + \frac{yj}{r} + \frac{z\underline{k}}{r} \\ &= (\cos\alpha)\underline{i} + (\cos\beta)\underline{j} + (\cos\gamma)\underline{k} \end{aligned} \qquad (12.3.3)$$

Thus, the coefficients of $\underline{i}, \underline{j}, \underline{k}$ in the rectangular resolution of a unit vector are the direction cosines of the vector.

Again, it is clear from Fig.12.9 and the theorem of Pythagoras that;

$$\begin{aligned} OR^2 &= OA^2 + AR^2 \\ &= OA^2 + AF^2 + FR^2 \\ \therefore \quad \underline{r}^2 &= x^2 + y^2 + z^2 \end{aligned} \qquad (12.3.4)$$

Hence, *the square of the modulus of a vector is equal to the sum of the squares of its rectangular components.* Thus, dividing (12.3.4) by $\underline{r}^2$, we have,

$$\frac{x^2}{\underline{r}^2} + \frac{y^2}{\underline{r}^2} + \frac{z^2}{\underline{r}^2} = 1$$

and using (12.3.2) we obtain;

$$\ell^2 + m^2 + n^2 = 1 \tag{12.3.5}$$

showing that the sum of the squares of direction cosines is equal to unity.

Now, let $R_1, R_2, R_3, ...$, be points (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), ..., respectively, and $\vec{OR}_1 = \underline{r}_1$, $\vec{OR}_2 = \underline{r}_2$, $\vec{OR}_3 = \underline{r}_3 ...$, then,

$$\begin{aligned} \underline{r}_1 &= x_1\underline{i} + y_1\underline{j} + z_1\underline{k}, \\ \underline{r}_2 &= x_2\underline{i} + y_2\underline{j} + z_2\underline{k}, \\ \underline{r}_3 &= x_3\underline{i} + y_3\underline{j} + z_3\underline{k}, ... \end{aligned}$$

hence, the sum of several vectors $\underline{r}_i$ can be expressed in the form:

$$\sum \underline{r}_i = \left(\sum x_i\right)\underline{i} + \left(\sum y_i\right)\underline{j} + \left(\sum z_i\right)k \tag{12.3.6}$$

where $\sum x_i$ is the resolute of $\sum \underline{r}_i$ in the direction of $\underline{i}$; and since the direction of $\underline{i}$ may be chosen arbitrarily, we may state the result:

The resolved part of the sum of any number of vectors in any direction is equal to the sum of the resolved parts or projections, of the individual vectors in the same direction.

On the other hand, from figure (12.10), the orthogonal projection of $\underline{r}$ on the plane YOZ is given as

$$\vec{OD} = Yj + Z\underline{k}$$

Thus, by (12.3.5), the projection of $\sum \underline{r}i$ on this plane i,

$$\sum \underline{r}_i = \left(\sum y_i\right)\underline{j} + \left(\sum z_i\right)\underline{k} = \sum(y_i j + z_i k)$$

Now, since the orientation of the triad $\underline{i}, \underline{j}, \underline{k}$, may be chosen arbitrarily, we may state the following remark.

Remark 39 *The resolute of a sum of vectors on any plane is equal to the sum of the resolute of the individual vectors on that plane.*

12.3.1 Position Vectors of Two Points in Space

We would recall that if $\underline{r}$ is regarded as the position vector of the point R relative to the origin O, then x, y, z are the resolute or rectangular Cartesian co-ordinates of R relative to the axes OX, OY, OZ. If $\underline{r}_1$ and $\underline{r}_2$ are the position vectors of two points R_1 and R_2 in space, relative to O;

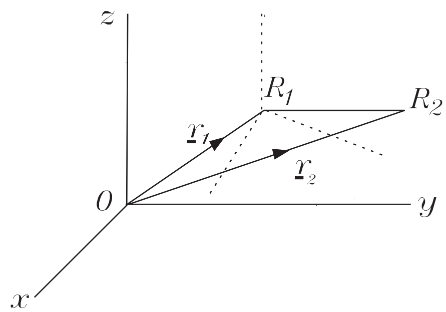

Figure 12.12: Position vectors of two points in space

then we have;

$$\vec{OR_1} = \underline{r}_1 = x_1\underline{i} + y_1 j + z_1\underline{k}$$

and

$$\vec{OR_2} = \underline{r}_2 = x_2\underline{i} + y_2\underline{j} + z_2\underline{k}$$

Changing the origin from O to R_1, the co-ordinate of R_2 becomes:

$$x_2 - x_1, y_2 - y_1, z_2 - z_1$$

such that

$$\vec{R_1R_2} = (x_2 - x_1)\underline{i} + (y_2 - y_1)j + (z_2 - z_1)\underline{k}$$

But

$$\begin{aligned} \vec{R_1R_2} &= \underline{r}_2 - \underline{r}_1 \\ &= (x_2 i + y_2 j + z_2 k) - (x_1 i + y_1 j + z_1 k) \end{aligned}$$

Consequently,

$$\vec{R_1R_2}^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 \qquad (12.3.7)$$

giving the modulus of $\underline{r}_2 - \underline{r}_1$, which is the square of the *distance between two points* in terms of their rectangular Cartesian co-ordinates.

Also, the direction cosines (12.3.2), $(\cos\alpha, \cos\beta, \cos\gamma)$ of (ℓ, m, n) of R_1R_2 are given by,

$$\frac{x_2 - x_1}{R_1R_2}; \ \frac{y_2 - y_1}{R_1R_2}; \ \frac{z_2 - z_1}{R_1R_2} \tag{12.3.8}$$

It is often convenient to denote a vector of the form: $x\underline{i} + y\underline{j} + z\underline{k}$ simply by (x, y, z). A point P, with this as a position vector, will be referred to as the point $P(x, y, z)$. Further, any unit vector can be expressed as ℓ, m, n, where ℓ, m, n are its direction cosines as in (12.3.8). For subtraction of vectors, we have;

$$\begin{aligned} \underline{r}_1 - \underline{r}_2 &= (x_1, y_1, z_1) - (x_2, y_2, z_2) \\ &= (x_1 - x_2, y_1 - y_2, z_1 - z_2) \end{aligned} \tag{12.3.9}$$

and for multiplication by a scalar, we obtain

$$\alpha(x, y, z) = (\alpha x, \alpha y, \alpha z)$$

α being a scalar. The modulus of the vector (x_1, x_2, x_3) is given as,

$$x = \sqrt{(x_1^2 + x_2^2 + x_3^2)}$$

and the unit vector in this direction is

$$\frac{(x_1, x_2, x_3)}{x}$$

Also, *the position vector of the point R, which divides the join of $X(x_1, x_2, x_3)$ and $Y(y_1, y_2, y_3)$ in the ratio of $a : b$, is*

$$\underline{r} = \left(\frac{ax_1 + by_1}{a + b}, \ \frac{ax_2 + by_2}{a + b}, \ \frac{ax_3 + by_3}{a + b}\right); \ (a + b) \neq 0 \tag{12.3.10}$$

It is observed that, in figure 12.12, X, Y are two points and $\underline{x}, \underline{y}$ their position vectors relative to the origin O. Then the position

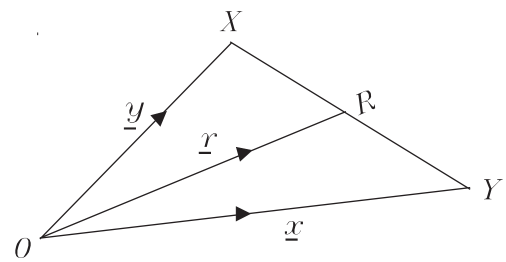

Figure 12.13: Position vector of a point which divides a line joining two vectors

vector of the point R which divides xy in the ratio $a : b$, may be expressed in terms of $\underline{x}$ and $\underline{y}$. For since,

$$a\vec{XR} = b\vec{RY}$$

it follows that;

$$\underline{r} = \frac{a\underline{x} + b\underline{y}}{a+b}; \quad (a + b \neq 0) \qquad (12.3.11)$$

where $\underline{x} = X(x_1, x_2, x_3)$, $\underline{y} = Y(y_1, y_2, y_3)$ which is equivalent to (12.3.10). The reasoning holds whether the ratio $a : b$ is positive or negative. In the latter case R is outside the segment XY.

For the particular case in which $a = b$, the above formula gives $\frac{1}{2}(\underline{x} + \underline{y})$ for the position vector of the midpoint of XY. Also (12.3.11) is equivalent to

$$a\vec{OX} + b\vec{OY} = (a + b)\vec{OR} \qquad (12.3.12)$$

where R is the mid-point dividing XY in the ratio $a : b$. This form of result is often useful.

Further, we can write (12.3.11) in the form:

$$(a + b)\underline{r} - a\underline{x} - b\underline{y} = 0 \qquad (12.3.13)$$

the sum of the coefficients of $\underline{r}$, $\underline{x}$, and $\underline{y}$ is zero. If R is distinct from X and Y, and at finite distance from them, non of the coefficients in (12.3.13) would be zero.

For three distinct collinear points R, X, Y there exists scalars α, β, γ, different from zero, such that,

$$\alpha\underline{r} + \beta\underline{x} + \gamma\underline{y} = 0; \quad \alpha + \beta + \gamma = 0 \qquad (12.3.14)$$

Conversely, when these relations hold, the three points are collinear.

12.3.2 Angle between Two Vectors

Let A and B be two points in space whose position vectors relative to the origin O, are $\underline{a}$ and $\underline{b}$, and coordinates (x_1, y_1, z_1) and (x_2, y_2, z_2) respectively.

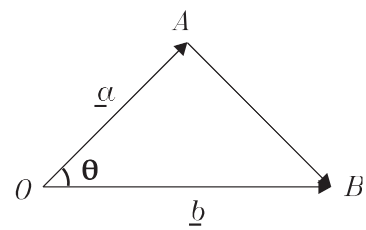

Figure 12.14: Angle between two vectors

Thus,

$$\begin{aligned} \underline{a} &= \vec{OA} = x_1 i + y_1 j + z_1 k, \\ \underline{b} &= \vec{OB} = x_2 i + y_2 j + z_2 \underline{k} \end{aligned}$$

Hence,

$$\vec{AB} = \underline{b} - \underline{a} = (x_2 - x_1)i + (y_2 - y_1)j + (z_2 - z_1)\underline{k}$$

or

$$AB = |\underline{b} - \underline{a}| = \{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2\}^{\frac{1}{2}}$$

Now, if θ is the angle between $\underline{a}$ and $\underline{b}$ (Figure 12.13), and using the cosine formula:

$$AB^2 = OA^2 + OB^2 - 2OA \cdot OB \cos\theta$$

or

$$|\underline{b} - \underline{a}|^2 = \underline{a}^2 + \underline{b}^2 - 2\underline{a}.\underline{b}\cos\theta$$

Hence,

$$\cos\theta = \frac{\underline{a}^2 + \underline{b}^2 - |\underline{b} - \underline{a}|^2}{2\underline{a} \cdot \underline{b}}$$

where

$$\underline{a}^2 = x_1^2 + y_1^2 + z_1^2;\ \underline{b}^2 = x_2^2 + y_2^2 + z_2^2$$

and we have,

$$\cos\theta = \frac{\sum x_1^2 + \sum x_2^2 - \sum (x_2 - x_1)^2}{2\underline{ab}} = \frac{x_1 x_2 + y_1 y_2 + z_1 z_2}{\underline{ab}} \tag{12.3.15}$$

If (ℓ_1, m_1, n_1) and (ℓ_2, m_2, n_2) be the direction cosines of $\vec{OA}$ and $\vec{OB}$ respectively, we recall that in (12.3.2);

$$\begin{aligned} \ell_1 &= \frac{x_1}{\underline{a}};\ m_1 = \frac{y_1}{\underline{a}};\ n_1 = \frac{z_1}{\underline{a}} \\ \ell_2 &= \frac{x_2}{\underline{b}};\ m_2 = \frac{y_2}{\underline{b}};\ n_2 = \frac{z_2}{\underline{b}} \end{aligned}$$

Consequently,

$$\cos\theta = \frac{x_1 x_2}{\underline{ab}} + \frac{y_1 y_2}{\underline{ab}} + \frac{z_1 z_2}{\underline{ab}} = \ell_1 \ell_2 + m_1 m_2 + n_1 n_2 \tag{12.3.16}$$

When $\cos\theta$ is known, $\sin\theta$ or $\tan\theta$ can easily be found.

Example 12.3.1

$\vec{OA}$ is a position vector of magnitude $8m$, and $\vec{OB}$ is a position vector of magnitude $10.4m$, inclined at an angle of $60°$ to $\vec{OA}$. Find the magnitude of the sum or resultant of these position vectors.

Solution

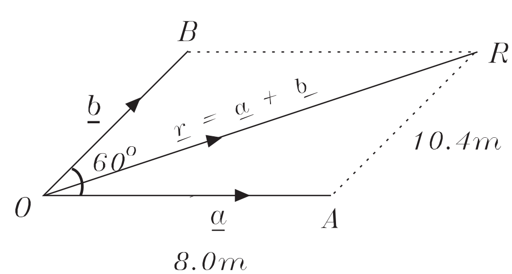

Figure 12.15:

(i) By calculation, using above Figure 12.15, and applying the cosine rule,

$$\begin{aligned}
OR^2 &= OA^2 + OB^2 - 2 \times OA \times OB \times \cos(OAR) \\
&= OA^2 + OB^2 + 2.OA.OB.\cos(AOB) \\
&= 8^2 + 10.4^2 + 2 \times 8 \times 10.4 \times \cos 60^o \\
&= 255.36 \\
\therefore \quad OR &= \underline{r} \\
&= |\underline{a} + \underline{b}| \\
&= \sqrt{255.36} \\
&\simeq 15.98m
\end{aligned}$$

(ii) By scale drawing – in order to find the magnitude of the vector $\underline{r}$, we draw to scale the above Figure 12.15; and to obtain reasonable accuracy, we select a scale that gives us a fairly large diagram. You may carry out the drawing to cross check your solution with what we have above.

Example 12.3.2

Consider a fisherman in a boat drifting powerlessly on Lagos Lagoon. Suppose that there is a current with velocity of $0.6ms^{-1}$ flowing due south, and a wind with velocity of $4.6ms^{-1}$ blowing from the east. For simplicity, we assume that there are no waves, and that the motion of the boat is not influenced by any factors other than the current and the wind mentioned above. Find the magnitude and direction of the boat's drift?

Solution

Let $\underline{w}, \underline{c}$ represent the velocities of the wind and ocean current respectively,

Since $\underline{w}$ and $\underline{c}$ are perpendicular, using Pythagoras Theorem;

$$\begin{aligned}
|\underline{w} + \underline{c}|^2 &= \underline{w}^2 + \underline{c}^2 \\
&= 0.6^2 + 4.6^2 \\
\therefore \quad |\underline{w} + \underline{c}| &= \sqrt{21.52} \simeq 4.64ms^{-1}
\end{aligned}$$

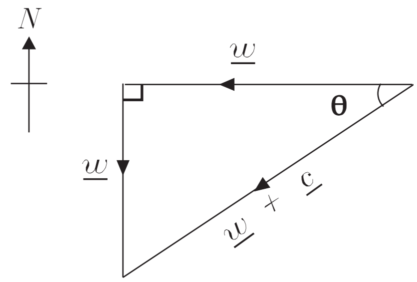

Figure 12.16:

Also,

$$\theta = arctan(0.6/4.64) = 7.368^\circ \simeq 7.37^\circ$$

Therefore the direction of the boat is $S7.37^\circ E$.

Alternatively, the problem could also be solved by scale drawing.

Example 12.3.3

The vector $\underline{x}$ represents a displacement of $6km$ on a bearing of 040°, and the vector $\underline{y}$ represents a displacement of $20km$ on a bearing of 310°. Find the magnitude and direction of the displacement represented by $2\underline{x} - \underline{y}$?

Solution

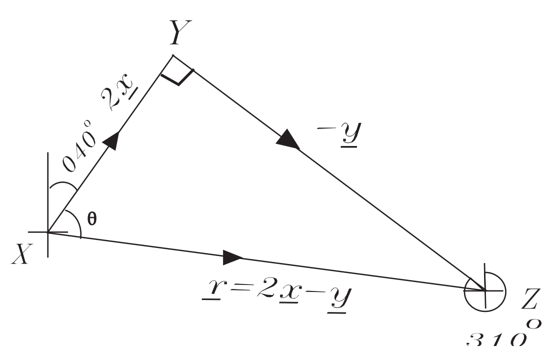

Figure 12.17:

Using Pythagoras Theorem,

$$\begin{aligned} \underline{r}^2 &= (2\underline{x})^2 + (-\underline{y})^2 \\ &= 4\underline{x}^2 + \underline{y}^2 \\ &= 4(6^2) + 20^2 \\ \therefore \quad \underline{r} &\simeq 23.32km \end{aligned}$$

Since the negative vector affects the direction and not the magnitude,

$$\tan\theta = \frac{\underline{y}}{2\underline{x}} = \frac{20}{12}$$

and so,

$$\theta = 59.04$$

Hence, $2\underline{x} - \underline{y}$ represents a displacement of $23.32km$ on a bearing of $99.04°$.

Example 12.3.4

Given that $\underline{a} = -i + 2j - 2k$; $\underline{b} = 3i + 4j - 12k$, find the:

(i) magnitude of $\underline{a}$ and $\underline{b}$

(ii) unit vector in the direction of $\underline{a}, \underline{b}$

(iii) direction cosines of $\underline{a}$

(iv) angle between $\underline{a}$ and $\underline{b}$

(v) angle between $3\underline{a} - 2\underline{b}$ and $2\underline{a} + \underline{b}$

Solution

(i)

$$|\underline{a}| = \sqrt{\{(-1)^2 + (2)^2 + (-2)^2\}} = \sqrt{9} = 3$$

Similarly,

$$|\underline{b}| = \sqrt{[(3)^2 + (4)^2 + (-12)^2]} = 13$$

(ii)

$$e_{\underline{a}} = \frac{\underline{a}}{a} = \frac{-i+2j-2k}{3} = -\frac{1}{3}i + \frac{2}{3}j - \frac{2}{3}k$$

and

$$e_{\underline{b}} = \frac{\underline{b}}{b} = \frac{3i+4j-12k}{13} = \frac{3}{13}i + \frac{4}{13}j - \frac{12}{13}k$$

(iii)

$$\frac{\underline{a}}{a} = -\frac{1}{a}i + \frac{2}{a}j - \frac{2}{a}k = -\frac{1}{3}i + \frac{2}{3}j - \frac{2}{3}k$$

$$\therefore \qquad \cos\alpha = -\frac{1}{3}, \quad \cos\beta = \frac{2}{3} \text{ and } \cos\gamma = \frac{-2}{3}$$

(iv) Since

$$e_{\underline{b}} = \frac{3}{13}i + \frac{4}{13}j - \frac{12}{13}k$$

and the direction cosines of $\underline{b}$ are $\frac{3}{13}$, $\frac{4}{13}$, and $\frac{-12}{13}$. If the angle between $\underline{a}$ and $\underline{b}$ is θ, then with direction cosines of $\underline{a}$ and $\underline{b}$ we obtain from (12.3.16);

$$\begin{aligned}
\cos\theta &= \ell_1\ell_2 + m_1m_2 + n_1n_2 \\
&= (-\frac{1}{3})(\frac{3}{13}) + (\frac{2}{3})(\frac{4}{13}) + (-\frac{2}{3})(-\frac{12}{13}) \\
&= 41.96^\circ
\end{aligned}$$

(v)

$$\begin{aligned}
3\underline{a} - 2\underline{b} &= 3(-i+2j-2k) - 2(3i+4j-12k) \\
&= (-3i+6j-6k) - (6i+8j-24k) \\
&= i(-3-6) + j(6-8) + k(-6+24) \\
&= -9i - 2j + 18k
\end{aligned}$$

and

$$\begin{aligned}
2\underline{a} + \underline{b} &= 2(-i+2j-2k) + (3i+4j-12k) \\
&= i(-2+3) + j(4+4) + k(-4-12) \\
&= i + 8j - 16k
\end{aligned}$$

Now, let $\underline{u} = 3\underline{a} - 2\underline{b}$ and $\underline{v} = 2\underline{a} + \underline{b}$ then,

$$\begin{aligned} e_u &= \frac{\underline{u}}{|u|} \\ &= \frac{-9i - 2j + 18k}{\sqrt{[(-9)^2 + (-2)^2 + (18)^2]}} \\ &= \frac{-9i - 2j + 18k}{\sqrt{409}} \end{aligned}$$

and so,

$$\begin{aligned} e_v &= \frac{\underline{v}}{|v|} \\ &= \frac{i + 8j - 16k}{\sqrt{[1 + 8^2 + (-16)^2]}} \\ &= \frac{i + 8j - 16k}{\sqrt{321}} \end{aligned}$$

Now, let the angle between $3\underline{a} - 2\underline{b}$ and $2\underline{a} + \underline{b}$ be ϕ. Thus,

$$\begin{aligned} \cos\phi &= \ell_1\ell_2 + m_1m_2 + n_1n_2 \\ &= \frac{1}{\sqrt{409}} \times \frac{1}{\sqrt{321}}[-9 - 16 - 288] \\ \therefore \quad \phi &= \cos^{-1}\left[\frac{-313}{\sqrt{409}.\sqrt{321}}\right] \\ &\simeq 149.75^\circ \end{aligned}$$

12.3.3 Concurrent Forces

Let a number of forces be acting at a point, and be represented by line of vectors which are the sides of a polygon. Suppose $\vec{OA}$, $\vec{AB}$, $\vec{BC}$, $\vec{CD}$, $\vec{DE}$, ... represented by the vectors F_1, F_2... respectively, and with lines of action concurrent at a point O; the single force represented by:

$$\underline{R} = \underline{F}_1 + \underline{F}_2 + \underline{F}_3 + ... = \sum \underline{F} \tag{12.3.17}$$

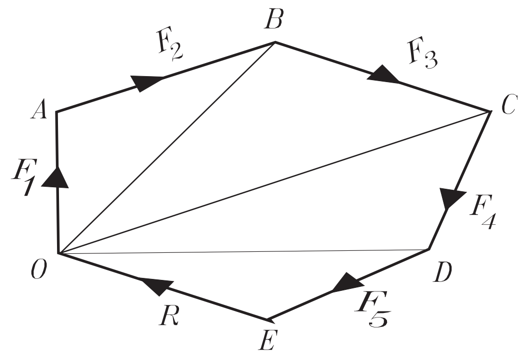

Figure 12.18: Concurrent forces represented by a polygon vector

and acting though the same point O, is dynamically equivalent to the system of forces, and is called their *resultant*. The vector $\underline{R}$ is determined by the *vector polygon*, and will not in general be closed, nor will it be plane unless the forces are coplanar. If $\vec{OA}$ is the first vector and $\vec{DE}$ the last, then $\vec{OE}$ is the resultant

$$\underline{R} = \sum \underline{F}$$

If the vector sum is equal to zero, or $\underline{R} = 0$, the polygon is closed, and the forces are in equilibrium; and since the resultant vanishes, the sums of the components of the several forces in any three non-coplanar or perpendicular directions vanish separately and conversely. This establishes the necessary and sufficient conditions of equilibrium of a number of concurrent forces.

If three forces acting at a point are in equilibrium, the closed polygon reduces to a triangle, and we arrive at an important theorem on Triangle of Forces:

Lami's Theorem 12.3.1. *If three concurrent forces, $\underline{F}_1, \underline{F}_2, \underline{F}_3$, acting at a point are represented by $\vec{AB}, \vec{BC}, \vec{CA}$ of triangle ABC, and are in equilibrium, they are coplanar, and each is proportional to the sine of the angle between the other two.*

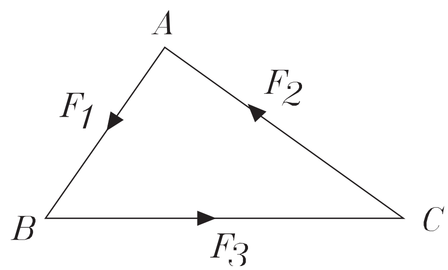

Figure 12.19: Three concurrent forces acting at a point

Thus,

$$\frac{BC}{\sin A} = \frac{CA}{\sin B} = \frac{AB}{\sin C} \tag{12.3.18}$$

or

$$\frac{F_1}{\sin A} = \frac{F_2}{\sin B} = \frac{F_3}{\sin C} \tag{12.3.19}$$

Example 12.3.21

P, Q, R, and S are the points with coordinates $(2, 1), (1, -2), (3, 3)$ and $(2, 0)$ respectively. Localized line vectors are defined to correspond to free vectors, so that, AB is defined by $\vec{AB}$. Find the sums: (a) $AB + CD$, (b) $AC + BD$, (c) $AC + AD$.

Give your answers in each of the following forms:

(i) $\alpha \underline{i} + \beta \underline{j}$ acting along a straight line whose equation is given: $y = mx + c$.

(ii) The magnitude of the sum, and the angle between its line of action and the x-axis, measured anticlockwise form Ox.

Solution

Let $\underline{i}$ and $\underline{j}$ be vectors along the x-axis and the y-axis respectively. So that

$$\vec{AB} = x\underline{i} + y\underline{j}$$

But

$$A(x_1, y_1) = A(2, 1) \text{ and } B(x_2, y_2) = B(1, -2)$$

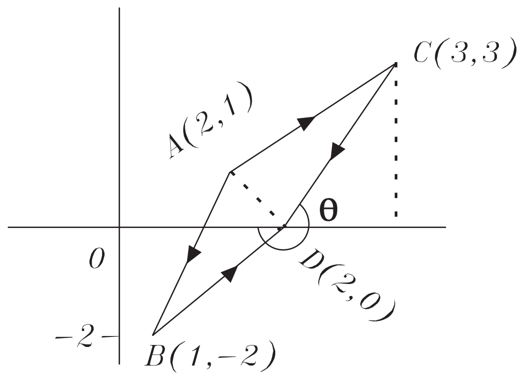

Figure 12.20:

Thus,

$$x = x_2 - x_1 = 1 - 2 = -1$$

and

$$y = y_2 - y_1 = -2 - 1 = -3$$

So that,

$$\vec{AB} = xi + yj = -i - 3j$$

Similarly,

$$\vec{CD} = x\underline{i} + y\underline{j}$$

But

$$X(x_3, y_3) = C(3,3) \text{ and } D(x_4, y_4) = D(2,0)$$

i.e.,

$$x = x_4 - x_3 = 2 - 3 = -1$$

and

$$y = y_4 - y_3 = 0 - 3 = -3$$

Thus,

$$\vec{CD} = -i - 3j$$

$$\begin{aligned} \therefore \qquad \vec{AB} + \vec{CD} &= (-i - 3j) + (-i - 3j) \\ &= -2i - 6j \end{aligned}$$

(i) The equation of the line AB is

$$y = mx + c$$

where $m = \frac{-3}{-1} = 3$ and $c = -5$
Thus,

$$y = 3x - 5 \qquad (i)$$

Similarly, the equation of the line CD is given as;

$$y = 3x - 6 \qquad (ii)$$

Adding (i) and (ii) we obtain;

$$\begin{aligned} 2y &= 6x - 11 \\ \therefore \quad y &= 3x - \frac{11}{2} \end{aligned}$$

(ii) The magnitude of the sum $\vec{AB} + \vec{CD}$ is:

$$\begin{aligned} |AD + CD| &= \sqrt{(-2)^2 + (-6)^2} \\ &= \sqrt{40} \\ &= 6.32 \end{aligned}$$

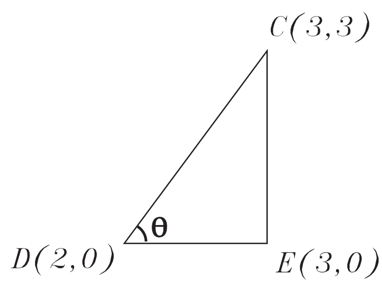

Figure 12.21:

Thus, the angle between the line of action and the x-axis, measured anticlockwise is

$$\phi = 180^o + \theta$$

where from figure 12.21 we have,

$$\begin{aligned} \theta &= \tan^{-1}\left(\tfrac{3}{1}\right) \\ &= 71.57^\circ \end{aligned}$$

Therefore, $\phi = 180 + 71.57° \simeq 251.57°$.
The solution to the other parts of the example is left as exercise.

12.4 Scalar or Dot Product

Suppose the magnitudes of the vectors $\underline{a}$ and $\underline{b}$ are a and b respectively. Then, we define the *scalar product* $\underline{a}.\underline{b}$ of $\underline{a}$ and $\underline{b}$ to be the scalar quantity $ab\cos\theta$, where θ is the angle between these vectors. The quantity $\underline{a}.\underline{b}$ is also called the inner product or dot product of two vectors, and is conventionally pronounced "a dot b'. Thus,

$$\underline{a}.\underline{b} = ab\cos\theta \tag{12.4.1}$$

and

$$\cos\theta = \frac{\underline{a}.\underline{b}}{|\underline{a}||\underline{b}|} \tag{12.4.2}$$

Example 12.4.1

Two position vectors $\underline{a}$ and $\underline{b}$ have the same length units and their scalar product is -2. Find the angle between them?

Solution

$$\underline{a}.\underline{b} = -2$$

Thus,

$$ab\cos\theta = -2$$

where θ is the required angle.
i.e.

$$16\cos\theta = -2$$

Since $a = 4$ and $b = 4$, we have;

$$\begin{aligned} \cos\theta &= \frac{-1}{8} \\ \text{or} \quad \theta &= 97.18° \end{aligned}$$

Example 12.4.2

Two vectors $\underline{p}$ and $\underline{q}$ have the same magnitude. The angle between them is 60°, and their scalar product is 6. Find their magnitude?

Solution

Suppose the vectors have length x.
Then,

$$\underline{p}.\underline{q} = pq\cos\theta$$

i.e.,

$$\begin{aligned} 6 &= x^2\cos 60^\circ \\ \text{Thus,} \qquad x &= \sqrt{\frac{6}{\cos 60}}, \text{ since } x \geq 0 \\ \text{and so} \qquad x &\simeq 3.464 \end{aligned}$$

12.4.1 Properties of Scalar Product

We will now establish a number of properties which hold for the scalar product that we have already defined. Let $\underline{a}$ and $\underline{b}$ be two vectors then the following is satisfied:

(i) $\underline{a}.\underline{b} = \underline{b}.\underline{a}$ (Commutative law)
This follows from $\underline{a}.\underline{b} = ab\cos\theta = ba\cos\theta = \underline{b}.\underline{a}$

(ii) $\underline{a}.(\alpha\underline{b}) = \alpha(\underline{a}.\underline{b});\ \forall\alpha \in R$ (Associative law)

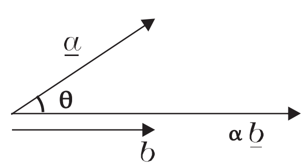

Figure 12.22: Associative properties of scalar product

We observe that,

$$\begin{aligned} \alpha(\underline{a}.\underline{b}) &= \alpha ab\cos\theta \\ \underline{a}.(\alpha\underline{b}) &= \alpha ab\cos\theta \end{aligned}$$

(iii) $\underline{a}.(\underline{c}+\underline{d}) = \underline{a}.\underline{c} + \underline{a}.\underline{d}$ (Distributive law)

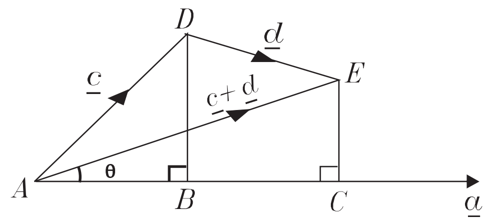

Figure 12.23: Distributive properties of scalar product

From Figure 12.23:

$$\begin{aligned} \underline{a}.(\underline{c}+\underline{d}) &= a.|c+d|\cos\theta \\ &= a.(AC) \\ &= a(AB+BC) \\ &= a.AB + a.BC \\ &= \underline{a}.\underline{c} + \underline{a}.\underline{d} \end{aligned}$$

This law might be expressed in words, as the projection of $\underline{c}+\underline{d}$ on $\underline{a}$, equals the sum of the projections of $\underline{c}$ and of $\underline{d}$ on $\underline{a}$.

Special Cases of Scalar Product

(i) If $\underline{a} = \begin{cases} \underline{b} = 1 : \underline{a}.\underline{b} \text{ gives the cosine of the angle between the} \\ \text{directions of } \underline{a} \text{ and } \underline{b} \\ 0 : \underline{a}.\underline{b} = 0.\underline{b} = 0 \\ \underline{b} : \underline{a}.\underline{b} = \underline{a}.\underline{a} = a^2 \end{cases}$

(ii) If $\theta = \begin{cases} 0 : \underline{a} \text{ and } \underline{b} \text{ have the same direction and } \underline{a}.\underline{b} = ab\cos\theta = ab \\ \pi : \underline{a} \text{ and } \underline{b} \text{ are directly opposed, and } \underline{a}.\underline{b} = ab\cos\pi = -ab \\ \frac{\pi}{2} : \underline{a} \text{ and } \underline{b} \text{ are perpendicular, and } \underline{a}.\underline{b} = ab\cos\frac{\pi}{2} = 0 \end{cases}$

(iii) If $\underline{a}.\underline{b} = 0 : \underline{a} = 0$ and/or $\underline{b} = 0$ and/or $\underline{a}$ and $\underline{b}$ are perpendicular.

(iv) If $\underline{a} = \underline{b} = i : \underline{a}.\underline{b} = \underline{i}.\underline{i} = 1 \times 1 \times \cos 0 = 1$,
Similarly $\underline{j}.\underline{j} = \underline{k}.\underline{k} = 1$.

(v) If $\underline{a} = i.\underline{b} = j$: $\underline{a}.\underline{b} = \underline{i}.\underline{j} = 1 \times 1 \times \cos\frac{\pi}{2} = 0$
Similarly $\underline{j}.\underline{k} = \underline{k}.\underline{i} = 0$ and $\underline{j}.\underline{i} = \underline{k}.\underline{j} = \underline{i}.\underline{k} = 0$

12.4.2 Orthogonality

Suppose we have two perpendicular vectors $\underline{a}$ and $\underline{b}$, then;

$$\underline{a}.\underline{b} = ab\cos 90^\circ = 0$$

i.e. the scalar product of two perpendicular vectors is zero.

Suppose, conversely, that we have two non zero vectors $\underline{a}$ and $\underline{b}$ whose scalar product is zero. Then,

$$ab\cos\theta = 0$$

where θ is the angle between $\underline{a}$ and $\underline{b}$; a and b are non-zero, and so $\cos\theta = 0$, and hence $\theta = 90^\circ$, i.e. the vectors are perpendicular.

Clearly, we have shown that two non-zero vectors $\underline{a}$ and $\underline{b}$ are perpendicular if and only if their scalar product is zero. We call such vectors *orthogonal*. Consequently, if $\underline{a}$ and $\underline{b}$ are non-zero; $\underline{a}$ and $\underline{b}$ are orthogonal if and only if $\underline{a}.\underline{b} = 0$.

For example, the unit vector $\underline{i}$ and $\underline{j}$ in the directions of the coordinate axes Ox and Oy are orthogonal, and

$$\underline{i}.\underline{j} = 0$$

Similarly, in the three-dimensional case,

$$\underline{i}.\underline{j} = \underline{j}.\underline{k} = \underline{k}.\underline{i} = 0$$

The three vectors $\underline{i}$, $\underline{j}$ and $\underline{k}$ are called *mutually orthogonal* because every pair is orthogonal.

These vectors are not just orthogonal; they are also unit vectors since,

$$\underline{i}.\underline{i} = \underline{j}.\underline{j} = \underline{k}.\underline{k} = 1$$

Thus, a pair of orthogonal unit vectors is said to be *orthonormal*. Consequently, A set such as $\{\underline{i}, \underline{j}, \underline{k}\}$, of mutually orthogonal vectors is called an *orthonormal set*

12.5 Scalar Product in $\mathbb{R}^3$

We may talk of the scalar product of two vectors $\underline{a}$ and $\underline{b}$ as;

$$\underline{a}.\underline{b} = ab\cos\theta$$

without specifying which type of vector we are dealing with. The exception is the case of column vectors, i.e. members of $\mathbb{R}^3$. We combine the various properties of scalar products of position vectors to justify the definition of $\underline{a}.\underline{b}$; for $\underline{a}, \underline{b} \in \underline{\mathbb{R}}^3$.

If

$$\begin{aligned} \underline{a} &= a_1\underline{i} + a_2\underline{j} + a_3\underline{k} \\ \text{and} \qquad \underline{b} &= b_1.\underline{i} + b_2\underline{j} + b_3\underline{k} \end{aligned}$$

with column vectors $\underline{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\underline{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ respectively.

Thus, we define the scalar product $\underline{a}.\underline{b}$ as:

$$\begin{aligned} \underline{a}.\underline{b} &= (a_1\underline{i} + a_2\underline{j} + a_3\underline{k}).(b_1\underline{i} + b_2\underline{j} + b_3\underline{k}) \\ &= a_1b_1\underline{i}.\underline{i} + a_1b_2\underline{i}.\underline{j} + a_1b_3\underline{i}.\underline{k} + a_2b_1\underline{j}.\underline{j} + b_2b_2\underline{j}.\underline{j} \\ &\quad + a_2b_3\underline{j}.\underline{k} + a_3b_1\underline{k}.\underline{i} + a_3b_2\underline{k}.\underline{i} + a_3b_2\underline{k}.\underline{k} \\ &= a_1b_1 + a_2b_2 + a_3b_3 \text{ (using orthogonality)} \end{aligned}$$

In particular,

$$\underline{a}.\underline{a} = a_1^2 + a_2^2 + a_3^2 = |\underline{a}|^2$$

in agreement with our earlier definition of $|\underline{a}|$.

This is an important result and states that:

The scalar product of two vectors is equal to the sum of the products of their corresponding rectangular coordinates.

If $\underline{a}$ and $\underline{b}$ are inclined at an angle θ, then

$$\cos\theta = \frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{\{a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2)\}}} \tag{12.5.1}$$

Example 12.5.1

Find the scalar product of the vectors (2,1,1) and (3,2,-1) and determine the cosine of the angle between them.

Solution

$$\begin{aligned}(2i+j+k).(3i+2j-k) &= (2)(3)+(1)(3)+(1)(-1)\\ &= 6+3-1=8\end{aligned}$$

If θ is the required angle, then;

$$\begin{aligned}\cos\theta &= \frac{(2)(3)+(1)(3)+(1)(-1)}{\sqrt{\{(2^2+1^2+1^2).(3^2+2^2+(-1)^2\}}}\\ &= \frac{8}{\sqrt{(6(15))}}\\ &= \frac{8}{\sqrt{(90)}}\\ &= \frac{8}{3\sqrt{10}}\\ &= \frac{8\sqrt{10}}{30}\\ &= \frac{4\sqrt{10}}{15}\end{aligned}$$

Example 12.5.2

P, Q and R are the points with Cartesian coordinates $(2,3,4)$, $(3,5,1)$ and $(3,-2,-1)$ respectively. Find angle $Q\hat{P}R$.

Solution

The vector $\underline{u}$ for $\vec{PQ}$ is:

$$\begin{aligned}\underline{u} &= \underline{q}-\underline{p}\\ &= (3\underline{i}+5\underline{j}+\underline{k})-(2\underline{i}+3\underline{j}+4\underline{k})\\ &= \underline{i}(3-2)+\underline{j}(5-3)+\underline{k}(1-4)\\ &= \underline{i}+2\underline{j}-3\underline{k}\\ \therefore \quad |\underline{u}| &= \sqrt{[1^2+2^2+(-3)^2]}\\ &= \sqrt{14}\end{aligned}$$

Similarly, the vector $\underline{v}$ for $\vec{PR}$ is:

$$\begin{aligned}
\underline{v} &= \underline{r} - \underline{p} \\
&= (3\underline{i} - 2\underline{j} - \underline{k}) - (2\underline{i} + 3\underline{j} + 4\underline{k}) \\
&= \underline{i}(3-2) + \underline{j}(-2-3) + \underline{k}(-1-4) \\
&= \underline{i} - 5\underline{j} - 5\underline{k} \\
\therefore \quad |\underline{v}| &= \sqrt{(1^2 + 5^2 + 5^2)} \\
&= \sqrt{51}
\end{aligned}$$

So that,

$$\underline{u}.\underline{v} = uv\cos\phi = \sqrt{14}\sqrt{51}\cos\phi$$

where ϕ is the angle between $\underline{u}$ and $\underline{v}$
Thus,

$$\begin{aligned}
\cos\phi &= \frac{\underline{u}.\underline{v}}{|\underline{u}||\underline{v}|} \\
&= \frac{(\underline{i} + 2\underline{j} - 3\underline{k}).(\underline{i} - 5\underline{j} - 5\underline{k})}{\sqrt{(14 \times 51)}} \\
&= \frac{(1)(1) + (2)(-5) + (-3)(-5)}{\sqrt{714}} \\
&= \frac{6}{\sqrt{714}}
\end{aligned}$$

Therefore $\phi = 77.02^\circ = Q\hat{P}R$.

Example 12.5.3

Show that $2\underline{i} + 3\underline{j} + \underline{k}$ and $3i - 2j$ are orthogonal, and hence find orthonormal vectors in their directions.

Solution

$$(2\underline{i} + 3\underline{j} + \underline{k}).(3\underline{i} + \underline{j}) = (2)(3) + (3)(-2) + 0 = 0$$

Therefore, the vectors are orthogonal; and their magnitudes are:

$$|2\underline{i} + 3\underline{j} + \underline{k}| = \sqrt{(2^2 + 3^2 + 1^2)} = \sqrt{14}$$

and

$$|3\underline{i} - 2\underline{j}| = \sqrt{(3^2 + (-2)^2)} = \sqrt{13}$$

Thus, orthonormal vectors in the directions of the given vectors are:

$$\frac{2}{\sqrt{14}}\underline{i} + \frac{3}{\sqrt{14}}\underline{j} + \frac{1}{\sqrt{14}}\underline{k} \text{ and } \frac{3}{\sqrt{13}}\underline{i} - \frac{2}{\sqrt{13}}\underline{j}$$

Note that the negatives of these are also orthonormal. In other words, we have two unit vectors, one in each sense, in any given direction.

Example 12.5.4

Prove the triangle formula by vector method:

$$c^2 = a^2 + b^2 - 2ab\cos C$$

Solution

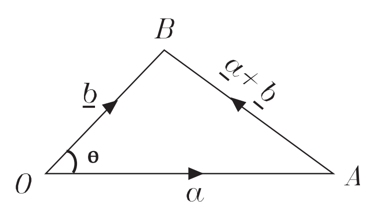

Figure 12.24:

If $|\underline{a} - \underline{b}| = \underline{c}$, since $\underline{a} - \underline{b}$ represents the third side of the triangle formed by $\underline{a}$ and $\underline{b}$ (Figure 12.18).
Then,

$$\begin{aligned} c^2 &= (\underline{a} - \underline{b}).(\underline{a} - \underline{b}) \\ &= \underline{a}.\underline{a} - 2\underline{a}.\underline{b} + \underline{b}.\underline{b} \\ \therefore \quad c^2 &= a^2 + b^2 - 2ab\cos c \end{aligned}$$

which is the well-known cosine formula for the triangle, reducing to:

$$c^2 = a^2 + b^2 \text{ (Pythagoras formula)}$$

when $\theta = \frac{\pi}{2}$, since $\cos\frac{\pi}{2} = 0$.

12.6 Vector or Cross Product

The *vector product* $\underline{a} \wedge \underline{b}$ or $\underline{a} \times \underline{b}$ (pronounced 'a cross b') of two vectors $\underline{a}$ and $\underline{b}$, whose directions are inclined at an angle θ, is the vector whose modulus is $ab \sin\theta$, and whose direction is perpendicular to both $\underline{a}$ and $\underline{b}$, being positive relative to a rotation from $\underline{a}$ to $\underline{b}$. In order not to be confused with the arithmetic operator sign of "×' i.e. multiplication, we shall restrict our use of 'cross product' to "$\underline{a} \wedge \underline{b}$' to mean "$a$ cross b' instead of "$a \times b$'.

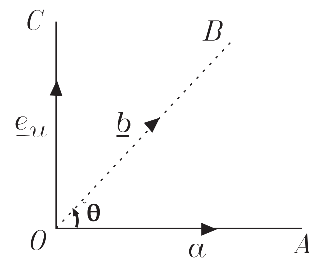

Figure 12.25: A unit vector perpendicular to a plane

Thus,

$$\underline{a} \wedge \underline{b} = ab \sin\theta \underline{e}_u \tag{12.6.1}$$

where $a = |\underline{a}|$, $b = |\underline{b}|$, and θ the smaller angle between the positive direction of $\underline{a}$ and $\underline{b}$, i.e. $\theta \leq 2\pi - \theta$; and $\underline{e}_u$ is a unit vector perpendicular to the plane $\underline{a}, \underline{b}$, having the same direction as the translation of a right-handed screw due to a direction from $\underline{a}$ to $\underline{b}$.

12.6.1 Properties of Vector Product

(i) Suppose $\underline{a}$ and $\underline{b}$ are position vectors, and that $\underline{a} \wedge \underline{b} = ab \sin\theta \underline{e}_u$. The right-handed screw turning from $\underline{b}$ to $\underline{a}$ defines the unit vector, $-\underline{e}_u$; and the magnitude of $\underline{b} \wedge \underline{a}$ is $ba \sin\theta$. Therefore,

$$\begin{aligned} \underline{b} \wedge \underline{a} &= (ba \sin\theta)(-\underline{e}_u) \\ &= -(ab \sin\theta)\underline{e}_u \end{aligned}$$

Hence,

$$\underline{b} \wedge \underline{a} = -\underline{a} \wedge \underline{b} \tag{12.6.2}$$

i.e vector multiplication is not commutative.

(ii) For two parallel vectors $\sin\theta = 0$, and their vector product vanishes. Thus, the condition of parallelism of two proper vectors $\underline{a}$ and $\underline{b}$ is:

$$\underline{a} \wedge \underline{b} = 0 \tag{12.6.3}$$

In particular, $\underline{r} \times \underline{r} = 0$ is true for all vectors.

(iii) If either vector $\underline{a}$ or $\underline{b}$ is multiplied by a scalar α; their vector product is multiplied by α.
Hence,

$$(\alpha\underline{a}) \wedge \underline{b} = \underline{a} \wedge (\alpha\underline{b}) = \alpha\underline{c}$$

where $\underline{a} \wedge \underline{b} = \underline{c}$

(iv) It is possible to express the length of $\underline{a} \times \underline{b}$ in terms of the scalar product. It can be shown that,

$$(\underline{a} \wedge \underline{b})^2 = \underline{a}^2\underline{b}^2 - (\underline{a}.\underline{b})^2$$

So that,

$$\begin{aligned} (\underline{a} \wedge \underline{b})^2 &= (\underline{a} \wedge \underline{b}).(\underline{a} \wedge \underline{b}) \\ &= (ab\sin\theta\underline{e}_u).(ab\sin\theta\underline{e}_u) \end{aligned} \tag{12.6.4}$$

where $\underline{e}_u$ is a unit vector perpendicular to the plane of $\underline{a}$ and $\underline{b}$ such that $\underline{a}.\underline{b}$ and $\underline{e}_u$ form a right-handed system.
Thus,

$$\begin{aligned} (\underline{a} \wedge \underline{b})^2 &= a^2b^2\sin^2\theta(\underline{e}_u.\underline{e}_u) \\ &= a^2b^2\sin^2\theta.1 \\ &= a^2b^2(1-\cos^2\theta) \\ &= a^2b^2 - a^2b^2\cos^2\theta \\ &= \underline{a}^2\underline{b}^2 - (\underline{a}.\underline{b})^2 \end{aligned} \tag{12.6.5}$$

(v) Distributive Law: $a \wedge (b + c) = \underline{a} \wedge \underline{b} + \underline{a} \wedge \underline{c}$.
To establish this law, we define the orthogonal projection $\underline{b}'$ of $\underline{b}$ onto the plane through O perpendicular to $\underline{a}$ as follows in Figure 12.26.

From the figure, $\underline{b}'$ is in the plane of $\underline{a}$ and $\underline{b}$; $\underline{b}'$ is in the plane through O, perpendicular to $\underline{a}$, the length of $\underline{b}'$ is $b\sin\theta$. The

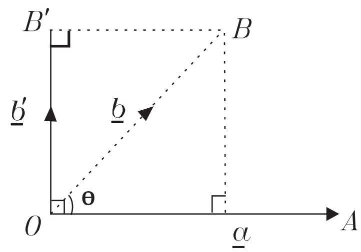

Figure 12.26: Orthogonal projection onto a plane through O

orthogonal projections $\underline{c}'$ of $\underline{c}$ of $\underline{c}$, and $\underline{d}'$ of $\underline{d} = \underline{b} + \underline{c}$ are also defined. Then,

$$\underline{a} \wedge \underline{b} = ab \sin\theta \underline{e}_u$$

where $\underline{e}_u$ is a unit vector perpendicular to the plane $\underline{a}$, $\underline{b}$ and b'. Thus,

$$\begin{aligned} \underline{a} \wedge \underline{b} &= (a \times b \sin\theta \times \sin\frac{1}{2}\pi)\underline{e}_u \\ &= (|\underline{a}| \times |\underline{b}'| \times \sin\frac{1}{2}\pi)\underline{e}_u \end{aligned}$$

In other words, $\underline{a} \wedge \underline{b} = \underline{a} \wedge \underline{b}'$, since $\frac{1}{2}\pi$ is the angle between $\underline{a}$ and $\underline{b}'$. Similarly, $\underline{a} \wedge \underline{c} = \underline{a} \wedge \underline{c}'$ and $\underline{a} \wedge \underline{d} = \underline{a} \wedge \underline{d}'$. Further, transform $\underline{b}'$ to $\underline{b}'$ as follows by rotating $\underline{b}'$ through one right-angle about $\underline{a}$, and enlarge by scale factor, $a = |\underline{a}|$. Then, $\underline{b}'$ is orthogonal to $\underline{a}$ and to $\underline{b}$ and $\underline{b}'$, and has magnitude:

$$|\underline{b}'| \wedge |\underline{a}| = ab \sin\theta$$

Consequently,

$$\underline{b}' = \underline{a} \wedge \underline{b}$$

Also, transform $\underline{c}'$ to $\underline{c}' = \underline{a} \wedge \underline{c}$ and $\underline{d}'$ to $\underline{d}' = \underline{a} \wedge \underline{d}$. Thus, we observe that the transformations of orthogonal projection, rotation, and enlargement all preserve, among other things, vector sums. Now,

$$\underline{d} = \underline{b} + \underline{c}$$

Thus,

$$\underline{d}' = \underline{b}' + \underline{c}'; \quad \textit{by first transformation}$$

Further,

$$\underline{d}' = \underline{b}' + \underline{c}'; \quad \textit{by second transformation}$$

Therefore,

$$\underline{a} \wedge \underline{d} = (\underline{a} \wedge \underline{d}) + (\underline{a} \wedge \underline{c})$$

and so

$$\underline{a} \wedge (\underline{b} + \underline{c}) = (\underline{a} \wedge \underline{b}) + (\underline{a} \wedge \underline{c})$$

as required.
Subsequently,

$$\begin{array}{rcl} (\underline{b} + \underline{c}) \wedge \underline{a} & = & -\underline{a} \wedge (\underline{b} + \underline{c}) \\ & = & -[(\underline{a} \wedge \underline{b}) + (\underline{a} \wedge \underline{c}] \\ & = & -(\underline{a} \wedge \underline{b}) - (\underline{a} \wedge \underline{c}) \\ & = & (\underline{b} \wedge \underline{a}) + (\underline{c} \wedge \underline{a}) \end{array}$$

which completes the assertion.

12.6.2 Cartesian Forms of the Vector Product

If $\underline{i}, \underline{j}, \underline{k}$ are mutually orthogonal vectors forming a right-handed system, it follows by definition;

$$\underline{i} \wedge \underline{i} = \underline{j} \wedge \underline{j} = \underline{k} \wedge \underline{k} = 0 \tag{12.6.6}$$

$$\left. \begin{array}{rcrcl} \underline{i} \wedge \underline{j} & = & -\underline{j} \wedge \underline{i} & = & \underline{k} \\ \underline{j} \wedge \underline{k} & = & -\underline{k} \wedge \underline{j} & = & \underline{i} \\ \underline{k} \wedge \underline{i} & = & -\underline{i} \wedge \underline{k} & = & \underline{j} \end{array} \right\} \tag{12.6.7}$$

i.e. by applying cyclic permutations of the line above. (Cyclic Permutations is discussed in Chapter 7 of this text). We may now use the laws of the previous paragraphs to express the vector products, $\underline{a} \wedge \underline{b}$ in terms of the Cartesian coordinates, $\underline{i}, \underline{j}, \underline{k}$ as follows:

$$(a_1\underline{i} + a_2\underline{j} + a_3\underline{k}) \wedge (b_1\underline{i} + b_2\underline{j} + b_3\underline{k})$$

$$\begin{aligned}
&= a_1b_1 i \wedge \underline{i} + a_1b_2\underline{i} \wedge \underline{j} + a_1b_3\underline{i} \wedge \underline{k} + a_2b_1\underline{j} \wedge \underline{i} \\
&\quad + a_2b_2\underline{j} \wedge \underline{j} + a_2b_3\underline{j} \wedge \underline{k} + a_3b_1\underline{k} \wedge \underline{i} + a_3b_2\underline{k} \wedge \underline{j} + a_3b_3\underline{k} \wedge \underline{k} \\
&= 0 + a_1b_2k - a_1b_3\underline{j} - a_2b_1\underline{k} + 0 + a_2b_3\underline{i} + a_3b_1\underline{j} - a_3b_2\underline{i} + 0 \\
&= (a_2b_3 - a_3b_2)\underline{i} + (a_3b_1 - a_1b_3)\underline{j} + (a_1b_2 - a_2b_1)\underline{k} \\
&= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}
\end{aligned} \tag{12.6.8}$$

This determinant is used to define the vector product of the corresponding row vectors;

$$\underline{a} = (a_1a_2a_3) \quad \text{and} \quad \underline{b} = (b_1b_2b_3)$$

by

$$\underline{a} \wedge \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

12.6.3 Sine of the Angle between two Vectors

Let

$$\underline{a} = a_1\underline{i} + a_2\underline{j} + a_3\underline{k}$$

and

$$\underline{b} = b_1\underline{i} + b_2\underline{j} + b_3\underline{k}$$

Thus,

$$|\underline{a}| = \sqrt{(a_1^2 + a_2^2 + a_3^2)}$$

and

$$|\underline{b}| = \sqrt{b_1^2 + b_2^2 + b_3^2)}$$

We know that,

$$\underline{a} \wedge \underline{b} = (a_2b_3 - a_3b_2)\underline{i} + (a_3b_1 - a_1b_3)\underline{j} + (a_1b_2 - a_2b_1)\underline{k} \tag{12.6.9}$$

If θ is the angle between the directions of $\underline{a}$ and $\underline{b}$, then the length of the vector on the L.H.S. of the above equation (12.6.9) is $ab \sin \theta$.

So that,

$$ab\sin\theta = \sqrt{(a_1^2 + a_2^2 + a_3^2)} \cdot \sqrt{(b_1^2 + b_2^2 + b_3^2)}\sin\theta$$

Similarly, the magnitude of the vector on the R.H.S. of (12.6.9) is given as:

$$\sqrt{[(a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2]}$$

Hence, equating these two length and simplifying, we obtain;

$$\sin^2\theta = \frac{(a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2}{(a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2)}$$

Suppose ℓ_1, m_1, n_1 and ℓ_2, m_2, n_2 are the direction cosines of $\underline{a}$ and $\underline{b}$ respectively, then;

$$\sin^2\theta = (m_1n_2 - m_2n_1)^2 + (n_1\ell_2 - n_2\ell_1)^2 + (\ell_1m_2 - \ell_2m_1)^2$$

It can be observed that if $\underline{b} = \underline{c} + n\underline{a}$ where n is any scalar, then;

$$\begin{aligned} \underline{a} \wedge \underline{b} &= \underline{a} \wedge (c + n\underline{a}) \\ &= \underline{a} \wedge \underline{c} + n(\underline{a} \wedge \underline{a}) \\ &= \underline{a} \wedge \underline{c} + n.0 \\ &= \underline{a} \wedge \underline{c} \end{aligned}$$

Conversely, if $\underline{a} \wedge \underline{b} = \underline{a} \wedge \underline{c}$ it does not follow that $\underline{b} = \underline{c}$, but that $\underline{b}$ differs from $\underline{c}$ by some vector parallel to $\underline{a}$, which may or may not be zero.

Example 12.6.1

Two vectors $\underline{a}$ and $\underline{b}$ are expressed in terms of vectors as follows:

$$\underline{a} = 3\underline{i} + \underline{j} + 2\underline{k}, \quad \underline{b} = 2\underline{i} - 2\underline{j} + 4\underline{k}$$

Determine the unit vector perpendicular to each of the vectors and hence, calculate the sine of the angle between the given vectors.

Solution

A vector perpendicular to the plane of $\underline{a}$ and $\underline{b}$ is the cross product given as:

$$\begin{aligned}
\underline{a} \wedge \underline{b} &= (3\underline{i} + \underline{j} + 2\underline{k}) \wedge (2\underline{i} - 2\underline{j} + 4\underline{k}) \\
&= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 3 & 1 & 2 \\ 2 & -2 & 4 \end{vmatrix} \\
&= (4+4)i - (12-4)j + (-6-2)k \\
&= 8i - 8j - 8k
\end{aligned}$$

and the length of this vector is:

$$|\underline{a} \wedge \underline{b}| = \sqrt{8^2 + 8^2 + 8^2)} = \sqrt{192} = 8\sqrt{3}$$

Hence, the required unit vector is obtained as:

$$\frac{1}{8\sqrt{3}}(8\underline{i} - 8\underline{j} - 8\underline{k}) = \frac{8}{8\sqrt{3}}(\underline{i} - \underline{j} - \underline{k}) = \frac{1}{\sqrt{3}}(1, -1, -1)$$

So that,

$$\begin{aligned}
\sin^2\theta &= \frac{8^2 + 8^2 + 8^2}{(3^2 + 1^2 + 2^2)(2^2 + 2^2 + 4^2)} \\
&= \frac{192}{14.24} \\
&= \frac{192}{336} \\
&= \frac{4}{7} \\
\therefore \qquad \sin\theta &= \frac{2}{\sqrt{7}}
\end{aligned}$$

Example 12.6.2

Find the unit vector perpendicular to both vectors, $(2, -1, 1)$ and $(3, 4, -1)$.

Solution

$\underline{a} \wedge \underline{b}$ is a vector perpendicular to both $\underline{a}$ and $\underline{b}$. A unit vector perpendicular to both therefore has as component, the direction cosines of their cross product. Thus,

$$\begin{aligned} (2,-1,1) \wedge (3,4,-1) &= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & -1 & 1 \\ 3 & 4 & -1 \end{vmatrix} \\ &= (1-4)\underline{i} - (-2-3)\underline{j} + (8+3)\underline{k} \\ &= -3\underline{i} + 5\underline{j} + 11\underline{k} \end{aligned}$$

and the magnitude of the vector is given as:

$$\sqrt{(3^2 + 5^2 + 11^2)} = \sqrt{155}$$

Therefore, the required unit vector is:

$$\frac{1}{\sqrt{155}}(-3i + 5j + 11k) = \frac{1}{\sqrt{155}}(-3, 5, 11)$$

Example 12.6.3

Find the condition of parallelism of vectors (a_1, a_2, a_3) and (b_1, b_2, b_3).

Solution

By (12.6.9), the required condition is:

$$(a_1, a_2, a_3) \wedge (b_1, b_2, b_3) = 0$$

This is equivalent to:

$$\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = 0$$

i.e.

$$(a_2b_3 - a_3b_2)\underline{i} + (a_3b_1 - a_1b_3)\underline{j} + (a_1b_2 - a_2b_1)\underline{k} = 0$$

Equating the coefficients of i, j, k on the two sides, we obtain;

$$a_2b_3 - a_3b_2 = 0; \quad \text{i.e.,} \quad \frac{a_2}{b_2} = \frac{a_3}{b_3}$$

$$a_3b_1 - a_1b_3 = 0; \quad \text{i.e.} \quad \frac{a_3}{b_3} = \frac{a_1}{b_1}$$

and

$$a_1b_2 - a_2b_1 = 0; \quad \text{i.e.} \quad \frac{a_1}{b_1} = \frac{a_2}{b_2}$$

Thus, the required condition is:

$$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$$

12.6.4 Some Applications of Vector Product

A plane area bounded by a closed curve without multiple points is capable of being represented by a vector provided that we distinguish between the two senses in which this curve may be described.

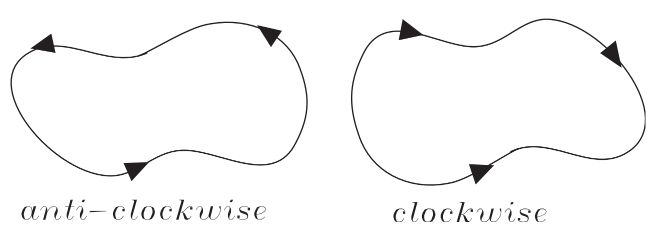

Figure 12.27: A plane area bounded by a closed curve without multiple points

Such an area is represented by a vector $\underline{a}$ defined as follows:

(i) The number of units of length of $\underline{a}$ is equal to the number of units of given area.

(ii) The direction of $\underline{a}$ is normal to the plane of the area.

(iii) The direction sense of $\underline{a}$ is such that the direction of the description of the area and the sense of $\underline{a}$ correspond to a right-handed screw.

Observe that if the direction of the area is reversed, then the sense of $\underline{a}$ will also be reversed.

Area of a Parallelogram

Consider the parallelogram $OADB$ whose sides OA and OB have the lengths and directions of $\underline{a}$ and $\underline{b}$ respectively, and suppose θ is the angle between these direction.

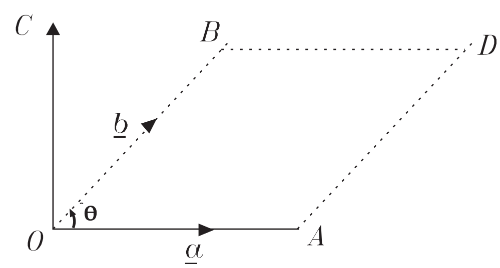

Figure 12.28: Parallelogram with vector area OADB

Thus, the area of the figure $OADB$ is $ab\sin\theta$ and the vector area $OADB$, whose boundary is described in the sense, is represented by

$$ab\sin\theta\underline{e}_u$$

where $\underline{e}_u$ is a unit vector perpendicular or normal to the plane of $\underline{a}$ and $\underline{b}$ such that $\underline{a}, \underline{b}$ and the sense of $\underline{e}_u$ form a right-handed system.

We know that $\underline{a} \wedge \underline{b} = ab\sin\theta\underline{e}_u$

Hence, it follows that the vector area $(OADB)$ is given as:

$$\text{Area } = ab\sin\theta = |\underline{a} \wedge \underline{b}| \tag{12.6.9}$$

The geometric representation of $\underline{a} \wedge \underline{b}$ is of such common occurrence and importance that it might be taken as the definition of the product. From this, the trigonometric definition follows.

Remark 40 *Since the vector area of triangle OAB (i.e.* $\triangle DOAB$*) is half the vector area of parallelogram* $OADB$*, we therefore define the area of triangle OAB as:*

$$\textit{Area } = \frac{1}{2}ab\sin\theta = \frac{1}{2}|\underline{a} \wedge \underline{b}| \tag{12.6.10}$$

12.7 Scalar Triple Product

We already know that the cross product $\underline{a} \wedge \underline{b}$ is itself a vector; now, we may form with it and a third vector $\underline{c}$ the scalar product $(\underline{a} \wedge \underline{b}).\underline{c}$, which is a scalar. Such products occur frequently, hence we shall find it useful to examine its properties. These properties are most easily deduced from its commonest geometrical interpretations. Consider the parallelepiped whose current edges OA, OB and OC have the lengths and directions of the vectors $\underline{a}$, $\underline{b}$ and $\underline{c}$ respectively.

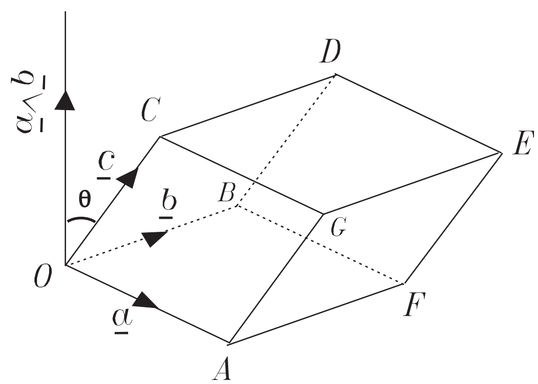

Figure 12.29: Scaler triple product: A parallelepiped

The vector $\underline{a}\wedge\underline{b}$, which may be denoted by $\vec{A}$, represents a vector whose magnitude is equal to the area A of the parallelogram $OAFB$ and whose direction is such that the vector $\underline{a}, \underline{b}$ and $\underline{a} \wedge \underline{b}$ is right-handed. Suppose θ is the angle between the directions of $\underline{e}_u$ and $\underline{c}$, and c is the length of the vector, $\underline{c}$ while $\underline{e}_u$ is a unit vector along the direction of $\underline{a} \wedge \underline{b}$, then;

$$\underline{a} \wedge \underline{b} = A\underline{e}_u$$

Hence, the triple product is given as:

$$\begin{aligned}(\underline{a} \wedge \underline{b}).\underline{c} &= (A\underline{e}_u).\underline{c} \\ &= A(\underline{e}_u.\underline{c}) \\ &= A(1.\underline{c}\cos\theta) \\ &= A\underline{c}\cos\theta = Ah = \pm V\end{aligned} \tag{12.7.1}$$

where v is the measure of the parallelepiped of which $\underline{a} \wedge \underline{b}$ is the base and $\underline{c}$ the height or edge. The triple product is positive if θ is acute; that is $\underline{a} \wedge \underline{b}$ and $\underline{c}$ lie on the same side of $\underline{ab}-$plane; but negative, if they lie on opposite sides. In other words, if the vector $\underline{a}$, $\underline{b}$, and $\underline{c}$ form a right-handed system, the scalar $(\underline{a} \wedge \underline{b}).c$ is positive; but if they form a left-handed system, it is negative. The cyclic order $\underline{a}, \underline{b}, \underline{c}$ is maintained in each of these. If however, that the order is changed, the sign of the product is also changed as well for instance, $\underline{b} \wedge \underline{c} = -\underline{c} \wedge \underline{b}$.
Thus,

$$\left.\begin{array}{rcl} \pm V & = & \underline{a}.(\underline{b} \wedge \underline{c}) = (\underline{b} \wedge \underline{c}).\underline{a} = -(\underline{c} \wedge \underline{b}).\underline{a} = -\underline{a}.(\underline{c} \wedge \underline{b}) \\ & = & \underline{b}.(\underline{c} \wedge \underline{a}) = (\underline{c} \wedge \underline{a}).\underline{b} = -(\underline{a} \wedge \underline{c}).\underline{b} = -\underline{b}.(\underline{a} \wedge \underline{c}) \\ & = & \underline{c}.(\underline{a} \wedge \underline{b}) = (\underline{a} \wedge \underline{b}).\underline{c} = -(\underline{b} \wedge \underline{a}).\underline{c} = -\underline{c}.(\underline{b} \wedge \underline{a}) \end{array}\right\} \tag{12.7.2}$$

Thus, the value of the product depends on the cyclic order of the factors, but it is independent of the position of the dot and cross. These may be interchanged at will. It is however customary to denote the above product by $[\underline{abc}]$ or $[\underline{a}, \underline{b}, \underline{c}]$, which indicates the three factors and their cyclic order. So that,

$$[\underline{abc}] = -[\underline{acb}] \tag{12.7.3}$$

12.7.1 Scalar Triple Product of Coplanar Vectors

If three vectors $\underline{a}$, $\underline{b}$ and $\underline{c}$ are coplanar, their scalar triple product is zero; $\underline{b} \wedge \underline{c}$ is then perpendicular to $\underline{a}$, and their dot product vanishes. Thus the vanishing of $[\underline{abc}]$ is the condition that the vectors should be coplanar. It follows that if two of the vectors are parallel, obviously, this condition is satisfied. In particular, if two of them are equal, the product is zero.

We shall consider an expression for the product $[\underline{abc}]$ in terms of rectangular components of the vectors. Thus, with the usual rotation;

$$\underline{b} \wedge \underline{c} = (b_2c_3 - b_3c_2, b_3c_1 - b_1c_3, b_1c_2 - b_2c_1)$$

and

$$\begin{aligned} \underline{a}.(\underline{b} \wedge \underline{c}) &= a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1) \\ &= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \end{aligned} \tag{12.7.4}$$

which is the well known expression given in analytical geometry for the volume of a parallelepiped; one of whose vertices is at the origin. More generally, if in terms of three non-coplanar vectors $\underline{\ell}, \underline{m}, \underline{n}$ we write,

$$\underline{a} = a_1\underline{\ell} + a_2\underline{m} + a_3\underline{n}$$

and so, it is easily established that;

$$[\underline{abc}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} [\underline{lmn}] \tag{12.7.5}$$

Clearly, the product $[\underline{ijk}]$, of three rectangular unit vectors is equal to unity.

Remark 41 *Since the distributive law holds for both scalar and vector products, it also holds for the scalar triple product; that is,*

$$[\underline{a}, \underline{b} + \underline{c}, \underline{d} + \underline{e}] = [\underline{abd}] + [\underline{abe}] + [\underline{acd}] + [\underline{ace}] \tag{12.7.6}$$

the cyclic order of the factors being preserved in each term.

Example 12.7.1

Find the volume of the parallelepiped whose three co-terminus edges are given by the vectors: $\underline{a} = (2, -3, 4)$, $\underline{b} = (1, 2, -1)$ and $\underline{c} = (3, -1, 2)$.

Solution

$$\begin{aligned} V &= (\underline{a} \wedge \underline{b}).\underline{c} \\ &= \begin{vmatrix} 2 & -3 & 4 \\ 1 & 2 & -1 \\ 3 & -1 & 2 \end{vmatrix} \\ &= 2(4-1) + 3(2+3) + 4(-1-6) \\ &= -7 \end{aligned}$$

i.e. $\underline{a}$, $\underline{b}$ and $\underline{c}$ form a left-handed system since the triple product, $(\underline{a} \wedge \underline{b}).\underline{c}$ is negative.

Example 12.7.2

Find the scalar triple product of $\underline{a}$, $\underline{b}$ and $\underline{c}$ and hence show that they are coplanar, for $\underline{a} = 3\underline{i} - 5\underline{j} + 6\underline{k}, \underline{b} = 2\underline{i} + 3\underline{j} - 2\underline{k}$, and $\underline{c} = -\underline{i} + 8\underline{j} - 8\underline{k}$.

Solution

By definition, if $\underline{a}$, $\underline{b}$ and $\underline{c}$ are coplanar, then $(\underline{a} \wedge \underline{b}).\underline{c} = 0$.
Thus,

$$\begin{aligned} (\underline{a} \wedge \underline{b}).\underline{c} &= \begin{vmatrix} 3 & -5 & 6 \\ 2 & 3 & -2 \\ -1 & 8 & -8 \end{vmatrix} \\ &= 3(-24+16) + 5(-16-2) + 6(16+3) \\ &= 0 \end{aligned}$$

and hence, the vectors $\underline{a}$, $\underline{b}$ and $\underline{c}$ are coplanar.

Chapter 13

Logic and Logical Applications

In Chapter 2, we discussed the mathematical representation of expressions, as well as the formation of equations and its evaluation for any given variable(s). For instance, we know that $4x = 8$; implies that $x = 2$ satisfies the equation, whereas $x \neq 2$ does not. Thus, the equation $4x = 8$ is valid at $x = 2$ only. Therefore, our understanding of the validity of mathematical expressions or statements in Chapter 2 will give us an insight into this chapter, as we shall be discussing symbolic representation of expressions/statements or arguments or propositions to ascertain their validity in logical constructions.

Ironically, the subject of this chapter, Logic, is also one of the most important branches of Philosophy. Hence, the Logician approaches the study of Logic as an art and science. One would therefore wonder at the inclusion of this chapter in a book of this nature. The justification is however borne out of the desire to relate the study of mathematics to real life situations at the slightest chance despite the elementary stage of the work in this book.

Logic may be defined as the normative science which investigates the principles of valid reasoning and correct inference, either from the general to the particular, *deductive logic*; or from the particular to the general, *inductive logic*. In addition, logic provides the techniques for testing the validity and invalidity of arguments and thus enhance the proficiency in reasoning. Reasoning here, imply the techniques of inferring conclusion from premises of arguments. Logicians are interested in correct reasoning, provided that adequate grounds for accepting the conclusions of an argument are laid.

Logic may be classified into two, viz:

(i) **Formal Logic:** The branch of logic which deals only with the formal structure of propositions and with the operations by which conclusions are deduced from them; the art of deductive reasoning.

(ii) **Symbolic Logic:** A development of formal logic in which the ambiguity of verbal propositions and of operations upon them is reduced to a minimum by the rigorous use of symbols each of which has only one referent within the given context. Also called *mathematical logic*; this is the area of logic that we shall be concerned with in this book.

13.1 Nature of Arguments

Literarily, argument is a process of reasoning to establish or refute a position by the use of evidence; or simply a reason offered for or against something. On the other hand, argument in mathematics is referred to as independent variable from which another quantity can be deduced or on which its calculation depends. Arguments contain a set of Premises and Conclusions.

Premise is a proposition laid down, proved, supposed, or assumed that serves as a ground for argument or for a conclusion. Examples of premise-indicators are: since; because; for; in as much as. On the other hand, *conclusion* is the absolute and necessary result of the admission of certain premises. Examples of conclusion-indicators are: *Thus; So; Therefore; It may be inferred; We may conclude; Hence; Consequently; It follows that.*

Consider the argument:

(i) All humans are mortal.

(ii) Socrates is human.

(iii) Therefore, Socrates is mortal.

Observe that (i) and (ii) are premises, while (iii) is the conclusion. This is an example of a *syllogistic argument* first started by Aristotle. But most arguments do not follow Aristotle's Syllogism.

As mentioned earlier, there are deductive and inductive types of arguments. In deductive argument, we reason from the general statements to particular statement i.e. we deduce or infer from the general statement to particular conclusion. For example;

All insect have six legs.

Mosquito is an insect.

$\therefore$ Mosquito has six legs.

In this type of argument, if the premises are true, the conclusion as a matter of necessity must be absolutely true or certain. The premises must provide necessary and sufficient grounds for accepting or affirming the conclusions.

On the other hand, inductive arguments proceeds from a particular instance to a general conclusion. Consider the argument:

The vulture seen today is white.

The one seen yesterday was white.

$\therefore$ All vultures are white.

In this type of argument, the premises only provide some grounds for the validity of the conclusion. The conclusion is never absolutely true or certain i.e. it only has a certain degree of probability.

Observe that in the course of these definitions, two words are important: Inference and Proposition.

Inference is a probable conclusion, toward which known facts, statements, or admissions point, but which they do not absolutely establish. However, we are not interested in the process of inference here, but in propositions which are the initial and end points of the process, and the relationships which exist between them. We shall explain proposition in the section that follows.

13.2 Symbolic (Mathematical) Logic

As discussed earlier, symbolic logic is the representation of the components of a proposition by symbols, to remove ambiguity and help to simplify the understanding in a given context. *Proposition* can therefore be defined as a statement in which something (the subject) is affirmed or denied in terms of something else (the predicate), the two being related usually by a copula. In the proposition; Grass is green and Grass is not red – grass in each case is the subject, and green and red are the predicates respectively. In other words, proposition is an idea expressed in a sentence. While a *sentence* on the other hand, is a word or a related group of words expressing a complete thought, whether as a statement of fact (declarative), a question (interrogative), a command (imperative) or an exclamation (exclamatory).

The truth or falsity of a statement or proposition is called its truth value; a proposition is either true or false, but not both. Some statements are *compound statements*, i.e. they are composed of sub-statements and various connectives. The fundamental property of a compound statement is that its truth value is completely determined by the truth values of its statements together with the way in which they are connected to form a compound statement. In symbolic logic, propositions are represented with alphabetical letters $A - Z$ (upper case or lowercase, with or without subscripts).

13.2.1 Truth Table

A truth table is a display of all possible combinations of truth values of a set of proposition or compound statement showing how each of such combinations will affect the value or validity of the entire component or compound statement. If the component or sub-statement in a compound statement are two, then the possible combinations of the truth values will be 2^2. Similarly, when the components are three, i.e. p, q, r, then the possible combinations will be 2^3 i.e. 8 possible values. Subsequently, for 4 components is 2^4, i.e. 16 possible values. Thus, the possible combinations of the truth values of any compound statement is given as:

$$\text{Truth values} = 2^n$$

where n is the number of components in any given compound statement.

For example, the truth values of two statements p and q in a compound statement are 2^2 and its truth table is given below:

Combinations	p	q
1	T	T
2	T	F
3	F	T
4	F	F

Table 13.1: Truth table combining 2 components

Similarly, for 3 components p, q and r the truth values is 2^3 and the truth table is as follows:

Combinations	p	q	r
1	T	T	T
2	T	T	F
3	T	F	T
4	T	F	F
5	F	T	T
6	F	T	F
7	F	F	T
8	F	F	F

Table 13.2: Truth table combining 3 components

13.2.2 Connectives

Apart from alphabetical letters used for the representation of various components in a compound statement or propositions, we also use other symbols which are important in removing ambiguities in our proposition and help to facilitate our thinking process. These symbols are referred to as connectives. Example of these symbols are:

(i) '$\wedge$' or '$\cdot$' as Conjunction

(ii) '$\vee$' as Disjunctive

(iii) '$\Rightarrow$' as Conditional

(iv) '$\Leftrightarrow$' as Bi-conditional

(v) '$\sim$' as Negation

(i) Conjunction, ($\wedge$ or $\cdot$)

Any two statements can be combined by the word 'and' to form a compound statement called the conjunction of the original statements. Symbolically, $p \wedge q$ denotes the conjunction of the statements p and q, read as 'p and q'; 'and' is the standard conjunction or connective while p, q are the conjuncts. The truth value of the proposition can be known through a truth table.

Consider the truth table of an arbitrary compound statement $p \wedge q$:

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Table 13.3: Truth table displaying conjunction

Here, the first line is a short way of saying that if p is true and q is true then $p \wedge q$ is true. The other lines have synonymous meaning. We regard this table as defining precisely the truth value of the compound statement $p \wedge q$ as a function of the truth values of p and of q. Observe that $p \wedge q$ is true only in the case that both statements are true.

Similar words that can be translated to mean 'and' are: also; nevertheless; although; however; both; while e.t.c

Example 13.1.1

Consider the following statement: Any person who is a Nigeria and eighteen years old is eligible to vote in any election in Nigeria. Using the truth table, establish the truth value of the proposition.

Solution

Using the truth table, we can establish the truth value of the above proposition as follows:

First, simplify by breaking the compound statement into various sub-statements or components and subsequently, assign symbols in the form of alphabetical letters to represent the sub-statements as follows;

p: Any person who is a Nigerian is eligible to vote in any election in Nigeria.

q: Any person who is eighteen years old is eligible to vote in any election in Nigeria.

The truth table is represented thus:

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Table 13.4

From the table above, $p \wedge q$ is true only when both statements are true as in line 1 and false or not correct otherwise in the remaining lines. In other words, for anyone to be eligible to vote in any election in Nigeria, the person must satisfy both conditions; i.e. a Nigerian and eighteen years old.

Example 13.1.2

Consider the following equations: $|-x| = x$ and $|-x| \neq -x$. Using a truth table, establish the truth value of the equations.

Solution

p: $|-x| = x$

q: $|-x| \neq -x$

The truth table is represented thus:

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Table 13.5

From the table, $p \wedge q$ is true only when p is true (T) and q is true (T), false otherwise.

(ii) Disjunction ($\vee$)

Disjunction is a compound statement formed by inserting the word 'or' between two simple propositions such that the resulting compound is true if one or the other of the component is true, and false otherwise.

Other words used to mean 'or' are: else; otherwise.

Example 13.2.3

Consider the statement: A brilliant student is either hardworking or punctual in class. Construct the truth table and hence establish the truth values.

Let the symbol p, q represents the sub-statements respectively: p: A brilliant student is hardworking q: A brilliant student is punctual in class

Symbolically it will be $p \vee q$; where p, q are called the disjuncts and '$\vee$' the disjunction. Thus truth table is represented thus:

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Table 13.6

From the truth table, $p \vee q$ is false only when both statements are false, while each of the other statements is true since at least one of its sub-statements is true.

Example 13.2.4

The root of the function, $f(x) = x^3 - 4x^2 - x + 4$ is either 1 or 3. Establish the truth values.

Solution

p: The root of the function, $f(x) = x^3 - 4x^2 - x + 4$ is 1

q: The root of the function, $f(x) = x^3 - 4x^2 - x + 4$ is 3

On evaluation, we observe that 1 is a root of $f(x)$, but 3 is not. The truth table is given thus:

p	q	$p \vee q$
T	T	T
T	F	T
F	T	F
F	F	T

Table 13.7

From the table, you may wonder why line 3 is false and line 4 is true. It is so because the second statement represented by q is not correct, and therefore it is true that 3 is not a root of $f(x)$.

(iii) Conditional, $(\Rightarrow)$

A conditional proposition is a compound statement produced with the operator, if... then. Symbolically, conditional statements are denoted by,

$$p \Rightarrow q$$

and it is read 'p implies q', 'p only if q', or 'if p, then q'.

Other words that may mean the implication are: Given A, it follows that q;Not p unless q; In case $p.q$; Given that p, q; Insofar as p, q; p leads to q; p is sufficient for q.

Example 13.2.5

Consider the statement: If Zaka is in the department of Mathematics, then he is a student. Establish the truth values.

Example 13.2.6

p: Zaka is in the department of Mathematics

q: Zaka is a student

Symbolically, we can write; if p, then q, i.e. $p \Rightarrow q$

The truth table of $p \Rightarrow q$ is represented thus:

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Table 13.8

From the truth table, the compound $p \Rightarrow q$ is false only when the antecedent is true and the consequent is false. In case p is false, the conditional $p \Rightarrow q$ is true regardless of the truth value of q.

(iv) Bi-conditional, $(\Leftrightarrow)$

A Bi-conditional is a compound proposition produced with the operator 'if and only if' such that the compound is true only when the sub-statements agree in truth value.

Other words used for Bi-conditional are: p is equivalent to q; p is necessary and sufficient for q; p just in case q; p just if q.

Example 13.2.7

You will receive your salary if and only if you have worked for the month.

Solution

p: You will receive your salary

q: You have worked for the month

Symbolically, p if and only if q, $p \Leftrightarrow q$.
The truth table is as follows:

p	q	$p \Leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Table 13.9

Note that from Table, the Bi-conditional $p \Leftrightarrow q$ is true when p and q have the same truth values and false otherwise.

Example 13.2.8

$x^2 - 1 = 0$ if and only if $x = 1$. Establish the truth values.

Solution

p: $x^2 - 1 = 0$

q: $x = 1$

The truth table is as follows:

p	q	$p \Leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Table 13.10

(v) Negation, $(\sim)$

Negation is a compound proposition called the negation of p, given any statement p which results when the word 'not' is inserted in p or can be formed by writing: 'It is false that ...' before p. Symbolically $\sim p$ denotes the negation of p (read not 'not p'). In truth table,

p	$\sim p$
T	F
F	T

Table 13.11

In other wards, if p is true then $\sim p$ is false, and if p is false then $\sim p$ is true. Thus the truth value of the negation of statement is always the opposite of the truth value of the original statement.

13.3 Logical Equivalence and Algebra of Propositions

Two propositions $P(p, q, ...)$ and $Q(p, q, ...)$ which have as component the same sub-statements, are said to be logically equivalent, or simply equivalent or equal, if and only if they have the same truth tables. Symbolically, we write,

$$P(p, q, ...) \equiv Q(p, q, ...)$$

Now, consider the truth table of the proposition, $\sim p \vee q$:

p	q	$\sim p$	$\sim p \vee q$	$p \Rightarrow q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Table 13.12

Observe that the last 2 columns of the truth Table are identical. Hence, $p \Rightarrow q$ is logically equivalent to the proposition $\sim p \vee q$:

$$p \Rightarrow q \equiv \sim p \vee q$$

Similarly, consider the conditional propositions which contains p and q:

$$q \Rightarrow p, \quad \sim p \Rightarrow \sim q \text{ and } \sim q \Rightarrow \sim p$$

These propositions are respectively called the *converse, inverse* and *contrapositive* of the proposition $p \Rightarrow q$. The truth tables of the four propositions are represented thus:

		Conditional	**Converse**	**Inverse**	**Contrapositive**
p	q	$p \Rightarrow q$	$q \Rightarrow p$	$\sim p \Rightarrow \sim q$	$\sim q \Rightarrow \sim p$
T	T	T	T	T	T
T	F	F	T	T	F
F	T	T	F	F	T
F	F	T	T	T	T

Table 13.13

The statements can be reversed making the consequence first, and that the conditional statement and its converse or inverse are not logically equivalent. On the other hand, a conditional statement and its contrapositive are seen to be logically equivalent. This is summarized below.

13.3.1 Converse of a Conditional Statement

If p and q are statements, then the converse of the conditional statement 'if p, then q' is the conditional statement 'if q, then p'. Symbolically, the converse of,

$$p \Rightarrow q \text{ is } q \Rightarrow p$$

Example: 13.3.1

If $x = 0$, then $x(x-2) = 0$. The converse of the conditional statement is:

If $x(x-2) = 0$, then $x = 0$

Observe that $x(x-2) = 0$ has two values i.e. $x = 0$ and $x = 2$ which does not agree with the conditional statement, hence;

$$p \Rightarrow q \equiv q \Rightarrow p$$

i.e. the conditional statement is not logically equivalent to its converse.

Corollary 13.3.1. *A conditional statement $p \Rightarrow q$ and its $p \Rightarrow q$ and its contrapositive $\sim q \Rightarrow \sim p$ are logically equivalent. The proof can be understood from the truth table in table 13.13.*

Further Examples

Example 13.3.2

Construct the truth table for the proposition $\sim (p \wedge \sim q)$.

Solution

p	q	$\sim q$	$p \wedge \sim q$	$\sim (p \wedge \sim q)$
T	T	F	F	T
T	F	T	T	F
F	T	F	F	T
F	F	T	F	T

Table 13.14

Example 13.3.3

To extend the generalization one more step, let us consider three simple statement p, q and r given by,

p : The applicant should be considered for the position of Treasurer
q : The applicant should be considered for the position of a Financial Accountant
r : The applicant has a criminal record

These three sub-statements can be expressed in a compound statement as follows:

The applicant should be considered for the position of Treasurer or Financial Accountant if and only if he has no criminal record.

Symbolically, we can write;

$$(p \vee q) \Leftrightarrow \sim r$$

Following the procedure used in previous sections, we use the truth table to illustrate this example:

Logical Possibilities	p	q	r	$p \vee q$	$\sim r$	$(p \vee q) \longrightarrow \sim r$
1	T	T	T	T	F	F
2	T	T	F	T	T	T
3	T	F	T	T	F	F
4	T	F	F	T	T	T
5	F	T	T	T	F	F
6	F	T	F	T	T	T
7	F	F	T	F	F	T
8	F	F	F	F	T	F

Table 13.15

Example 13.3.4

Determine the truth values of each of the following statement: If $x + 3 = 6$, then $x = 3$.

Solution

Let the sub-statements be given by,
$p \ : \ x + 3 = 6$
$q \ : \ x = 3$

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Table 13.16

Example 13.3.5

Verify that $\sim (p \Leftrightarrow q)$ is logically equivalent or otherwise to $p \Leftrightarrow \sim q) \vee \sim p$.

Solution

p	q	$\sim p$	$\sim q$	$p \Leftrightarrow q$	$\sim (p \Leftrightarrow \sim q)$	$p \Leftrightarrow \sim q$	$(p \Leftrightarrow \sim q) \vee \sim p$
T	T	F	F	T	F	F	F
T	F	F	T	F	T	T	T
F	T	T	F	F	T	T	T
F	F	T	T	T	F	F	T

$$\therefore \quad \sim (p \Leftrightarrow q) \not\equiv (p \Leftrightarrow \sim q) \vee \sim p$$

Table 13.17

Example 13.3.6

Prove that the following propositions are equivalent,
$(p \wedge q) \Rightarrow r$ and $(p \Rightarrow r) \vee (q \Rightarrow r)$.

Solution

Logical Possibilities	p	q	r	$p \wedge q$	$(p \wedge q) \Rightarrow r$	$p \Rightarrow r$	$q \Rightarrow r$	$a \vee b$
1	T	T	T	T	T	T	T	T
2	T	T	F	T	F	F	F	F
3	T	F	T	F	T	T	T	T
4	T	F	F	F	T	F	T	T
5	F	T	T	F	T	T	T	T
6	F	T	F	F	T	T	T	T
7	F	F	T	F	T	T	F	T
8	F	F	F	F	T	T	T	T

$$\therefore \quad (p \wedge q) \Rightarrow r \equiv (p \Rightarrow r) \vee (q \Rightarrow r)$$

Table 13.18

where from table 13.18: $a = (p \Rightarrow r)$ and $b = (q \Rightarrow r)$

13.4 Tautologies and Contradictions

A proposition $P(p, q, r, ...)$ which is true for all logical possibilities i.e. contain only T in the last column of their truth tables is called a *tautology*. Similarly, a proposition $P(p, q, r, ...)$ which is false for all members of the set of logically possibilities i.e. contains only F in the last column of its truth table is called a *contradiction*. For example, a proposition 'p or not p', i.e., $p \vee \sim p$, is a tautology while the proposition 'p and not p', i.e. $p \wedge \sim p$, is a contradiction. This is verified by constructing their truth tables:

p	$\sim p$	$p \vee \sim p$
T	F	T
F	T	T

} Tautology

p	$\sim p$	$p \wedge \sim p$
T	F	F
F	T	F

} Contradiction

Table 13.19

Note that, the negation of a tautology is a contradiction since it is always false, and the negation of a contradiction is a tautology since it is always true.

Example 13.4.1

Prove that $[(p \Leftrightarrow q) \wedge q] \Rightarrow p$ is a tautology.

Solution

p	q	$p \Leftrightarrow q$	$(p \Leftrightarrow q) \wedge q$	$[(p \Leftrightarrow q) \wedge q] \Rightarrow p$
T	T	T	T	T
T	F	F	F	T
F	T	F	F	T
F	F	T	F	T

Table 13.20

$\therefore$ $[(p \Leftrightarrow q) \wedge q] \Rightarrow p$ is a tautology, since it is true for all the truth values of the variables p and q.

Example 13.4.2

Prove that the proposition $(p \wedge q) \wedge \sim (p \vee q)$ is a contradiction.

Solution

p	q	$p \wedge q$	$p \vee q$	$\sim (p \vee q)$	$(p \wedge q) \wedge \sim (p \vee q)$
T	T	T	T	F	F
T	F	F	T	F	F
F	T	F	T	F	F
F	F	F	F	T	F

Table 13.21

Observe that the truth value of $(p \wedge q) \wedge \sim (p \vee q)$ is F for all the truth values of p and q hence it is a contradiction.

13.5 Logical Implication and Consistency

Several questions with answers which can be determined by applying mathematics involve such concepts as counting, approximation, measurement, and numerical computations. Another class of questions have solutions which depend on a study of relationships rather than on quantitative analysis. Hence '*relation*' is an important mathematical concept. One relation has already been introduced, that is the *relation of equivalence* of a set of propositions. From Section 13.2, the propositions are equivalent if and only if the truth values of the propositions are the same. (An additional requirement is that the propositions must include the same sub-statements).

In this section, two more relations on sets of propositions will be considered. These relations are logical implication and consistency.

13.5.1 Logical Implication

A proposition $P(p, q, r, ...)$ is said to *logically imply* a proposition $Q(p, q, r, ...)$, if and only if, when proposition $p(p, q, r, ...)$ is true, $Q(p, q, r, ...)$ is true as well. Symbolically, we write;

$$P(p, q, r, ...) \Rightarrow Q(p, q, r, ...) \tag{13.5.1}$$

The reader may refer to Section 13.2.3 for the relation of equivalence, the assumption is made that propositions $P(p, q, r, ...)$ and $Q(p, q, r, ...)$ involve the same sub-statements.

Example 13.5.1

One Sunday afternoon in the cafeteria at Edo State University, three mathematics students, Ernest, Hassan and Magnus, overheard two female students, Akpome and Joyce. The argument involves proposed registration of elective courses in the current academic session. After considerable argument, it was clear that the position taken by Akpome can be summarized by the proposition:

Our GPA will improve, and we shall not register for electives.

Similarly, the position taken by Joyce can be summarized by the proposition thus:

If we register for electives, then our GPA will improve.

After they had left, Ernest, Hassan, and Magnus discussed the positions taken by Akpome and Joyce. Ernest said he believes that the two girls just do not understand one another and that their positions are the same; that is, Ernest believes that the propositions which summarizes the position taken by Akpome and Joyce are equivalent. Hassan, on the other hand does not agree with Ernest. Hassan, is of the opinion that Akpome and Joyce are in disagreement, but that the proposition of Akpome implies the proposition of Joyce. More so, Magnus took a different viewpoint from Ernest and Hassan and observed that the proposition of Joyce implies the proposition of Akpome.

The statement of the problem is somewhat complicated; there are five persons, each of whom has a position, and two different arguments. It will be instructive to try to decide which (if any) of the mathematics students – Ernest, Hassan, or Magnus – you agree with before reading what follows. The first task is to isolate the propositions which are crucial to the argument and label then with the usual notation, thus:

p: GPA will improve without electives.
q: Electives will be registered to improve GPA.

With this agreement, the position of Akpome is represented symbolically by,

$$p\wedge \sim q$$

Similarly, the position of Joyce is expressed symbolically by,

$$q \Rightarrow p$$

Reading the propositions that summarized their respective position to see that these expressions are true.

Symbolically, the truth tables for each of these propositions can be represented thus:

p	q	$\sim q$	$p\wedge \sim q$
T	T	F	F
T	F	T	T
F	T	F	F
F	F	T	F

(a)

p	q	$q \Rightarrow p$
T	T	T
T	F	T
F	T	F
F	F	T

(b)

Table 13.22

Table 13.22(a) and (b) represents the position of Akpome, and the position of Joyce respectively. The positions taken by Ernest, Hassan, and Magnus can be evaluated by use of the Tables (a) and (b).

First, Ernest had observed that the propositions of Akpome and Joyce were *equivalent*. But recall that two propositions can be equivalent if and only if they have the same truth values. Consequently, a glance at Table I and II shows that the truth values are not the same. In other words, Ernest is wrong!

Second, Hassan had observed that the proposition of Akpome *implies* the statement of Joyce. In other words, he agrees that what this means is that *in every case for which Akpome proposition is true Joyce's proposition is true as well*. (You may now wish to restate your stand in view of this agreement). Look at the tables. It would be observed that there is only one case when $p\wedge \sim q$ is true i.e. the second line), and $q \Rightarrow p$ is true for this case as well. Hassan is correct!

Third, Magnus is of the opinion that Hassan is not correct. If the word *implies* is to mean that when $q \Rightarrow p$ (Joyce's proposition) is true, then $p \wedge \sim q$ (Akpome's proposition) is true also, then $q \Rightarrow p$ does *not* imply $p \wedge \sim q$. In the tables, $q \Rightarrow p$ is true in the first case but $p \wedge \sim q$ is false! This situation also prevails in the fourth case, (but one such case is enough to illustrate that $q \Rightarrow p$ does not imply $p \wedge \sim q$). In a more elaborate form, we shall reconstruct the truth tables above by merging Tables I and II together:

p	q	$\sim q$	$p \wedge \sim q$	$q \Rightarrow p$	$(p \wedge \sim q) \Rightarrow (q \Rightarrow p)$	$a \Rightarrow b$
T	T	F	F	T	T	F
T	F	T	T	T	T	T
F	T	F	F	F	T	T
F	F	T	F	T	T	F

Table 13.23

where $a = (q \Rightarrow p)$ and $b = (p \wedge \sim q)$ in table 13.23.
It is observed that the 6th column from left i.e. $(p \wedge \sim q) \Rightarrow (q \Rightarrow p)$ has all its truth values as T, and hence is a tautology, it follows that the argument is valid, on the other hand, the last column is not a tautology, therefore the relation, $(q \Rightarrow p) \Rightarrow (p \wedge \sim q)$ is not valid.

Example 13.5.3

Show that the truth values of $p \Leftrightarrow q$ logically implies $p \Rightarrow q$ and not vice versa.

Solution

p	q	$p \Leftrightarrow q$	$p \Rightarrow q$	$(p \Leftrightarrow q) \Rightarrow (p \Rightarrow q)$	$(p \Rightarrow q) \Rightarrow (p \Leftrightarrow q)$
T	T	T	T	T	T
T	F	F	F	T	T
F	T	F	T	T	F
F	F	T	T	T	T

Table 13.24

Observe that $(p \Leftrightarrow q) \Rightarrow (p \Rightarrow q)$ is a tautology, therefore it is valid while, $(p \Rightarrow q) \Rightarrow (p \Leftrightarrow q)$ is not, hence it is not valid.

13.5.2 Consistency

The propositions $P(p, q, r, ...)$ and $Q(p, q, r, ...)$ are said to be *consistent* if and only if there is at least one logical possibility for which both propositions are true. Equivalently, $P(p, q, r, ...)$ and $Q(p, q, r, ...)$ are inconsistent if and only if for no logical possibility are both propositions true. However, the definition for consistency given for pairs of proposition can be extended to three or more. For example, propositions $P(p, q, r, ...)$, $Q(p, q, r, ...)$, and $R(p, q, r, ...)$ are consistent if and only if there is a logical possibility such that all three is true, otherwise, they are inconsistent.

Example 13.5.4

The sales manager Mr. Awanfi O. Adams of Cowbell Milk was upset over transportation schedules and was discussing the matter with Mr. Peter Udosen of the transport department. He said;

'*if products leave the factory on time, then the sales representative will meet their sales quotas*'.

Mr. Peter Udosen, the Head of the Transport Department, replied:

'*products leave the factory on time, and the sales representative do not meet their sales quotas*'.

One of the directors, who has brought the two men together to air their differences cuts in that their propositions are *inconsistent*. At that point, the adversaries demand that the director explain his remark. Consequently, he explained that products may not go out on time and that the sales representative may or may not meet their sales quotas but under no circumstances are the propositions of the two men true. The three men agreed that if both propositions are not true simultaneously, then they are *inconsistent*, and they decided to put the proposition of the Mr. Awanfi Adams and Mr. Peter Udosen to this test. (Is the director right?)

In what follows, symbols are given to the propositions involved, and the propositions are written in symbolic form i.e.,

p : products leave the factory on time

q : the sales representative meet their sales quotas

The proposition of the sales manager Mr. Awanfi Adams is represented thus: $p \Rightarrow q$
while, the proposition of Mr. Peter Udosen, head transport is given as, $p \wedge \sim q$. Thus, the truth tables are:

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Table 13.25(a)

p	q	$\sim q$	$p \wedge \sim q$
T	T	F	F
T	F	T	T
F	T	F	F
F	F	T	F

Table 13.25(b)

From the Tables 13.25 (a) and (b) respectively, the truth sets for these propositions have no element in common. What does this example indicate for the definition of consistency? Actually, the dispute was about *inconsistency*, with an agreement that inconsistency means that for no case are both propositions true. But, if inconsistency means that for no case are both propositions true, then *consistency* should mean that at least one case both propositions are true. It therefore means that the director is right and hence their propositions are inconsistent.

Example 13.5.5

Show that the propositions $p \Leftrightarrow \sim q$ and $p \Rightarrow q$ are consistent or not.

Solution

p	q	$\sim q$	$p \Leftrightarrow \sim q$
T	T	F	F
T	F	T	T
F	T	F	T
F	F	T	F

Table 13.26(a)

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Table 13.26(b)

From Tables 13.26(a) and (b), it is in line three that both propositions have the same truth value T, hence by the definition of consistency, there exist at least one case where both propositions are true, hence the proposition are consistent

13.6 Quantification Theory

In the previous sections of this chapter, we analyzed sentences and arguments, breaking them down into constituent simple statements, regarding these simple statements as the building blocks. By this means, we were able to discover something of what makes a valid argument. However, there are arguments which are not susceptible to such a treatment. For example, let us write down one of the example in Section 13.1

All women are selfish
Juliet is a woman
$\therefore$ Juliet is selfish

This, we would regard intuitively as an example of a valid argument, but if we try to symbolize the form of the argument as we did in the previous section, what we get is;

$$p, q, \therefore \quad r$$

According to these sections this is not a valid argument form.

Validity in this case depends not upon the relationships of the premises and conclusion as simple statements, but upon relationships between parts of the statements involved and upon the forms of the statements themselves. If we wished to make this clearer by finding a corresponding argument form', it would have to look like this:

All $X's$ are Y
Z is an X
$\therefore$ Z is Y

There are two points to be dealt with - first, the general nature of the premise All $X's$ are Y', and next, the use of symbols to represent parts of simple statements. These points correspond respectively to the ideas of *quantifier* and *predicate*. In English, every simple

statement has a subject and a predicate, each of which may consist of a single word, a short phrase or a whole clause. If we may put it crudely, the subject is that term about which something is affirmed or denied, and the predicate refers to a property, which the subject has.
However, from the first example in this section, the second premises, Juliet is a woman' can be considered as a singular sentence. It says that the individual, Juliet has the attribute of being a woman,

Juliet: Subject term

Woman: Predicate term

We already know that an equation whose expression do not contain variables is a proposition, which can be true or false; for example,

$$3 + 2 = 5 \text{ and } 3(5 + 2) = 20 + 1$$

are true propositions, while

$$2 + 3 - 4 = 5$$

is a false proposition. But if the expression contain variables, then the equation is a *predicate*, for example, the equations,

$$3x = -12, \quad 49 + 3b = 1, \quad x^2 = \frac{(6x + 24)}{3}$$

Only after numbers from the domain of definition of the equation are substituted for the variables, the predicate becomes a proposition, which may be true or false.

Symbolically, we shall denote predicate terms by capital letters $(A - Z)$, and subjects by small letters. In logic, we regard an individual not only as persons, but anything such as animals, nations, cities e.t.c.

Example 13.6.1

In each of the following statements, the subject is in boldface and the remainder is the predicate.

(i) **Juliet** is a woman

(ii) **I** read books

(iii) **The number whose square is -1** is not real

Solution

(i) $W(j)$ may stand for 'Juliet is a woman' where W is a predicate letter standing for 'is a woman' and j stands for 'Juliet'.

(ii) $B(i)$ may stand for 'I read books' in a similar way.

(iii) In this case, our predicate is a negation so we have a choice. Let R denote the predicate term 'is not real', so that the statement would have the form $R(c)$, where c represents 'the number whose square is -1'. Symbolically, the statement would be $(\sim R(c))$.

Suffice it to say that compound statements can also be translated into symbols in this way, just by symbolizing all the constituent simple statements.

We have just considered singular sentence; now what about statements such as 'All women are selfish'? We need something better than a subject-predicate analysis, because the meaning of the statement depends on the force of the word 'all'. Consider another example:

Every integer has a prime factor.

In ordinary mathematical symbolism, we might write this as:

For all x, if x is an integer, then x has a prime factor.
Using the kind of symbolic language just introduced, we could write this as:

For all x, $[I(x) \Rightarrow P(x)]$

where $I(x)$ denotes 'x is an integer' and $P(x)$ for 'x has a prime factor'.

Similarly, if we introduce predicate symbols W for is a woman and 'is selfish' as S then 'all women are selfish' may be written:

For all x, $[W(x) \Rightarrow S(x)]$.

The phrase 'for all x' is called a *universal quantifier* and is translated into symbols as '$(\forall x)$'. Note that when we write;

$$(\forall x)[W(x) \Rightarrow S(x)] \tag{13.6.1}$$

there is no assumption about the nature of the object x. The implication is asserted 'for every object x in the universe', if x is a woman, then x is selfish. In other words, for any x which is a woman, whether the particular x is selfish is irrelevant. The implication is true because its first part is false (refer to Section 13.5 for '$\Rightarrow$').

From our illustrations, the small letter x is called *individual variables*, and it is the indeterminate subjects. When they are used as above in statements starting with quantifiers, they are said to be *bound variables*.

The expression say $W(x)$ is called *propositional function*. A propositional function is an expression that contains individual variable and becomes a proposition when the individual variable is replaced by an individual constant. This process is called *instantiation*.

There is another kind of quantifier which is at first sight needed to translate ordinary English statements into symbols. Consider the sentence:

Some men are selfish.

A rephrasing of this would be 'there is at least one man who is selfish', or using a device similar to that above; there exists at least one object y such that y is a man and y is selfish. The phrase 'there exists at least one object y such that' is called an *existential quantifier*, and is translated into symbols as $(\exists y)$. The sentence may now be written:

$$(\exists y)(M(y) \wedge S(y)) \tag{13.6.2}$$

where $M(y)$ and $S(y)$ mean 'y is a man' and 'y is selfish' respectively.

More generally, if A is a symbol which denotes a predicate, then we can write meaningfully:

$$(\forall x)A(x) \text{ and } (\exists x)A(x) \tag{13.6.3}$$

The first means 'every object has the property determined by A', and the second means 'there is some object which has the property determined by A'.

Example 13.8.2

Translate into symbols:

(i) Not all birds can fly

(ii) Any fish can swim

(iii) Some students are lazy

(iv) There is a real number which is smaller than every other real number

Solution

(Note that there may be different approaches).

(i) $\sim (\forall x)(B(x) \Rightarrow F(x))$

(ii) $(\forall x)(F(x) \Rightarrow S(x))$

(iii) $(\exists x)(S(x) \wedge L(x))$

(iv) $(\exists x)(R(x) \wedge (\forall y)(R(y) \Rightarrow x \leq y))$

Observe that in (ii), 'Any fish can swim' means 'all fish can swim', and just as before this means 'for all x, if x is a fish, then x can swim'. $F(x)$ denotes 'x is a fish', and $S(x)$ 'x can swim'. On the other hand, in (iv) $(\exists x)$ denotes 'there exists a real number such that', where x denote '*a* real number' and y 'other real number', so that $R(x)$ denote 'x is a real number' whereas $(\forall y)$ and $R(y)$ denote 'every other real number' and y is other real number respectively.

Remark 42 *We shall find that these illustrate a common, but not universal pattern. The universal quantifier is very often followed by an implication, because a universal statement is most often of the form 'given any x, if it has property A then it also has property B'.*

While the existential quantifier is often followed by a conjunction, because an existential statement is most often of the form 'there exists an x with property A, which also has property B'.

If we may consider (i) further; it is obvious that this assertion is true, and we can justify it by the instances of Ostriches, Penguins or Peacock. Intuitively, we are justifying not all birds can fly by justifying 'there is a bird which cannot fly'. Here is an important connection between the two quantifiers, for a moments thought will convince one that the statement above have the same meaning. Let us translate them into symbols:

(i) $\sim (\forall x)(B(x) \Rightarrow F(x))$

(ii) $(\exists x)(B(x) \wedge \sim F(x))$
To compare these more closely, let us transform the first into;

(iii) $\sim (\forall x)(\sim B(x) \vee F(x))$, according to the rules of symbolic logic Section 13.1. It can further be transformed into;

(iv) $\sim (\forall x) \sim (B(x) \wedge \sim F(x))$. It is observed that (iv) has a similar form to (ii), but with $\sim (\forall x) \sim$ in place of $(\exists x)$

Consideration of examples like these enables us to see intuitively that the two sentences:

(i) It is not the case that all x's do not have the property p,

(ii) There exists an x which has property p, have the same meaning, whatever property p stands for.

Example 13.6.4

Translate each of the following statements into symbols, first using universal quantifier and next existential quantifier.

(i) All integers are real numbers

(ii) No knowledge is useless

(iii) Some real numbers are not integer

Solution

(i) $(\forall x)(I(x) \Rightarrow R(x))$ or $\sim (\exists x)(I(x) \wedge \sim R(x))$

(ii) $(\forall x)(K(x) \Rightarrow \sim U(x))$ or $\exists x K(x) \wedge \sim U(x)$

(iii) $\forall x)(R(x) \Rightarrow \sim I(x))$ or $(\exists x)(R(x) \wedge \sim I(x))$

(Note that other equivalent approach can be used).

Since we have known how to translate into symbols, how does this help in deciding the relationships between statements or the validity of arguments? It is not possible to extend the use of truth tables, because our sentences now are not truth functional. The use of variables and quantifiers means that the truth value of a sentence does not depend as it did before merely on the truth values of the component parts, nor do the parts always have truth values. In particular, it makes no sense to talk of the truth value of a part of a sentence containing a variable which has no quantifier, for instance, 'x is knowledge' or $K(x)$. However, the main result of quantification is that the whole propositional function becomes a proposition which is either true or false. It should be noted that the universal quantification of a propositional function is true if and only if its substitution instances are true. Also the existential quantification of a propositional function is true if it has at least one true substitution instance.

Remark 43 *If we accept that there is at least one individual, then every propositional function has at least one substitution instance which is either true or false. Given this assumption, if the universal quantification of a propositional function is true, then its existential quantification must be true as well.*

Suppose that individual variable 'x' ranges over a finite set of individuals, a set for example whose numbers are named by the individual constants a, b, c, d, and e. Then to say that '$(\forall x)A(x)$' is simply to say that;

$$A(a) \wedge A(b) \wedge A(c) \wedge A(d) \wedge A(e) \tag{13.6.4}$$

and so to say that '$(\exists x)A(x)$' is simply to say that,

$$A(a) \vee A(b) \vee A(c) \vee A(d) \vee A(e) \quad (13.6.5)$$

Indeed, for finite domains of individual variables, universally quantified general sentences expand into equivalent finite conjunctions of singular sentences and existentially quantified formulas expand into equivalent finite disjunctions of singular sentences. This of course, applies in general to only finite domains.

Example 13.6.5

If the universal set $U = \{1, 2, 3\}$, and $\forall$ be a universal quantifier, $\exists$ an existential quantifier. Determine the truth value of the following statement: $(\exists x)\forall y : x^2 < y + 1$, where $x, y \in U$ and $y \subseteq U$.

Solution

When $x = 1$:

y	$(\exists x = 1)\forall y : x^2 < y + 1$
1	$1^2 < 1 + 1 \equiv 1 < 2 : T$
2	$1^2 < 2 + 1 \equiv 1 < 3 : T$
3	$1^2 < 3 + 1 \equiv 1 < 4 : T$

When $x = 2$:

y	$(\exists x = 2)\forall y : x^2 < y + 1$
1	$2^2 < 1 + 1 \equiv 4 < 2 : F$
2	$2^2 < 2 + 1 \equiv 4 < 3 : F$
3	$2^2 < 3 + 1 \equiv 4 < 4 : F$

When $x = 3$:

y	$(\exists x = 3)\forall y : x^2 < y + 1$
1	$3^2 < 1 + 1 \equiv 9 < 2 : F$
2	$3^2 < 2 + 1 \equiv 9 < 3 : F$
3	$3^2 < 3 + 1 \equiv 9 < 4 : F$

In summary, we have;

y	$x = 1$	$x = 2$	$x = 3$
1	T	F	F
2	T	F	F
3	T	F	F

which gives:

x	y	$(\exists x)\forall y : x^2 < y + 1$
1	1	T
1	2	T
1	3	T
2	1	F
2	2	F
2	3	F
3	1	F
3	2	F
3	3	F

Table 13.27

13.7 Application of Logical Operations

One of the greatest achievement of mathematical logic is in its application to the development and construction of electrical switching circuits and logic gates. This distinguished feat has contributed tremendously to the development of computers and electronic circuitry which has revolutionized today into a digital world. The remaining part of this chapter and indeed this book shall be devoted to this fundamental aspect of mathematical logic.

13.7.1 Switching Circuits

We can relate the laws of propositions to the laws of switching circuits. When current flows in an electric circuit; it flows through a switch or switches. The circuit is said to be *closed* if current flows through the circuit otherwise when no current flows through the circuit it is said to be *open*. Similarly, we shall symbolize the switches by $P, Q, R, ...$ just as we did with statements in propositions.

In electrical circuit, we have two types of arrangement as follows:

Observe that the two switches P and Q in fig. 13.1a are in series. It follows that if switch P is closed and Q is open and

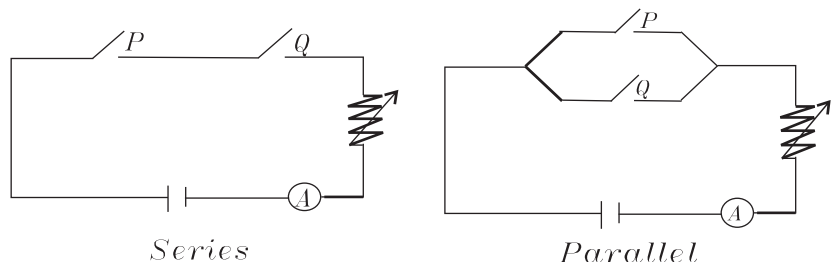

Figure 13.1: Electrical circuit in series and parallel

vice-versa, no current flows in the circuit. On the other hand, if both switches P and Q are closed, simultaneously, then current flows in the circuit. Consequently, in the series arrangement current flows in the circuit if and only if both switches are *closed*. The arrangement therefore corresponds to the conjunction in logic and it is represented symbolically by;

$$P \wedge Q \Leftrightarrow C \tag{13.7.1}$$

where C represents the proposition '*current flows*'.

In Figure 13.1(b) we say that the switches P and Q are arranged in parallel. In this arrangement, when P is closed current flows through P but not Q and otherwise. However, if both P and Q are closed, current flows through as well. It therefore follows that current flows through provided that one or the other or both of the switches P, Q is closed. In other words, this arrangement corresponds to the disjunction in logic and can be represented symbolically by,

$$P \vee Q \Leftrightarrow C \tag{13.7.2}$$

where C represents the proposition '*current flows*'.

In electrical circuits, we use binary digits '0 and 1' (instead of F and T as we have in logical statements) to represent '*no current flows' and 'current flows*' respectively. In other words, the binary digit '0' implies the circuit is open while 1 implies circuit is closed. A simpler notation which is often used in the algebra of switching circuits will be introduced thus:

We shall write,

$$\bar{P} \text{ instead of } \sim P$$
$$P + Q \text{ instead of } P \vee Q$$
$$PQ \text{ instead of } P \wedge Q$$

With these notations, we can compare the algebra of propositions and switching circuits and observe the similarities in the following:

Proposition	Neg. (Not)	Disj. (OR)	Conj. (AND)
Logic	$\sim P$	$P \vee Q$	$P \wedge Q$
Switching Circuits	$\bar{P}$	$P + Q$	PQ

Table 13.28: Comparison of algebra of propositions and switching circuits.

Suppose we have the statements:

$P =$ Switch P is closed

$Q =$ Switch Q is closed

$C =$ Current flows in the circuit

Then, for Figure 13.1(a) $P \wedge Q \Leftrightarrow C$ and Figure 13.1(b) $P \vee Q \Leftrightarrow C$ will become,

$$\begin{aligned} PQ &= 1 \\ P + Q &= 1 \\ P\bar{P} &= 0 \end{aligned}$$

i.e. expressing these statements in our new notation.

Furthermore, we examine what happens for the various cases of P, Q open or closed, if we construct a closure table. Such tables are truth tables except that we write 1 ('Switch closed' or 'current flows') instead of T (true) and write 0 ('switch open' or 'no current flows') instead of F (false). Figures 13.1(a) and (b) can be represented by the truth tables in Tables 13.29(a) and (b) respectively:

P	Q	PQ
1	1	1
1	0	0
0	1	0
0	0	0

(a)

P	Q	$P+Q$
1	1	1
1	0	1
0	1	1
0	0	0

(b)

Table 13.29: Closure table

From Tables 13.29(a) and (b) respectively, closure tables are exactly synonymous with the truth Table 13.2.1 and Table 13.2.2 obtained for $P \wedge Q$ and $P \vee Q$. Consequently, two switching circuits are said to be *equivalent* if and only if they have the same closure values.

In switching circuit algebra, we denote a switch by the variables $P, Q, R, S, \ldots$. However, if two switches, say P and Q open and close simultaneously we denote them by the same variable. Whereas if one is open when the other is closed and vice-versa we denote one by say P and the other by the negation of P i.e. $\bar{P}$. Parallel switches P and Q are denoted by $P + Q$, while switches in series by PQ. With these rules, we can build up a function for any circuit which involves sets of switches which may be connected either in series or in parallel.

Example 13.7.1

Draw a diagram of the circuit whose formula is given as:
$(P + Q)(P + R)$ and complete the corresponding closure table.

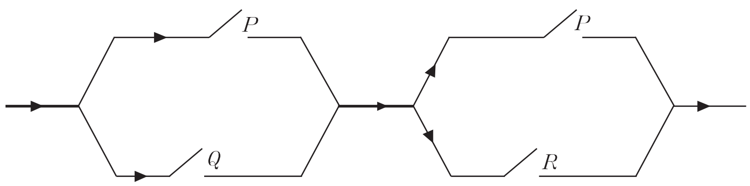

Figure 13.2:

Solution

The closure table is as follows:

P	Q	R	$P+Q$	$P+R$	$(P+Q)(P+R)$
1	1	1	1	1	1
1	1	0	1	1	1
1	0	1	1	1	1
1	0	0	1	1	1
0	1	1	1	1	1
0	1	0	1	0	0
0	0	1	0	1	0
0	0	0	0	0	0

Table 13.30

Example 13.7.2

Write down the formula that describes the following circuit and hence complete the closure table.

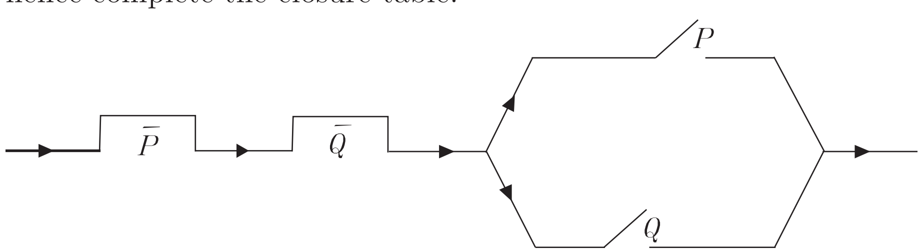

Figure 13.3:

Solution

The formula is $\bar{P}\bar{Q}(P+Q)$.
Closure table is as follows:

P	Q	$\bar{P}$	$\bar{Q}$	$\bar{P}\bar{Q}$	$P+Q$	$\bar{P}\bar{Q}(P+Q)$
1	1	0	0	0	1	0
1	0	0	1	0	1	0
0	1	1	0	0	1	0
0	0	1	1	1	0	0

Table 13.31

13.7.2 Logic Gates

A digital computer is a digital system that performs various computational tasks. The word digital implies that the information in the computer is represented by variables that take a limited number of discrete or quantified values. These values are processed internally by components that can maintain a limited number of discrete states. The decimal digits $0, 1, 2, ..., 9$, for example, provide 10 discrete values. In practice, digital computers function more reliably if only two states are used. Because of the physical restriction of components, and because human logic tends to be binary (i.e. True or False, Yes or No statements), digital components that are constrained to take discrete values are further constrained to take only two values 0 and 1, and are said to be binary.

Binary logic deals with binary variables and with operations that assume a logical meaning. Binary logic is used to describe, in algebraic or tabular form, the manipulation and processing of binary information. The manipulation of binary information is done by logic circuits called *gates*. Gates are blocks of hardware that produce signals of binary 1 or 0 when input logic requirements are satisfied. Various logic gates are commonly found in digital computer systems. Each gate has a distinct graphic symbol and its operation can be described by means of an algebraic function. The input-output relationship of the binary variables for each gate can be represented in tabular form in a truth table. The types of logical circuits used in digital computer may, in a broad sense, be divided into only two classes: *Combinational logic circuits* and *Sequential logic circuits*. We shall however, consider combinational logic circuits in our discussion here and interested readers in Sequential Logic Circuits may consult any Electrical Engineering book on circuit theory.

13.7.3 Combinational Logic Circuits

A combinational circuit is a connected arrangement of logic gates with a set of inputs and outputs. At any given time, the binary values of the outputs are a function of the combination of 1's and 0's at the inputs. A block diagram of a combinational circuit is depicted in Figure 13.3. The m input variables i.e. $I_1, I_2, ..., I_m$ come from an external source, while the n output variables i.e. $0_1, 0_2, ..., 0_n$, go to an external destination; and in between there is an interconnection of logic gates in which the state of inputs at any instant of time is determined completely by the state of the inputs at that instant.

A combinational circuit transforms binary information from the given input data to the required output data. This definition distinguishes the combinational circuit from a sequential circuit in which the state of the outputs at any instant of time depends not only on the inputs at that instant but also on the previous sequence of inputs which has occurred. Note that the size of m and n may vary widely from one circuit to another. You can have a case of several input lines to the circuit but only a single output line and this is very often the case; in other cases the number of output lines may be many and may even exceed the number of input lines to the combinational circuit.

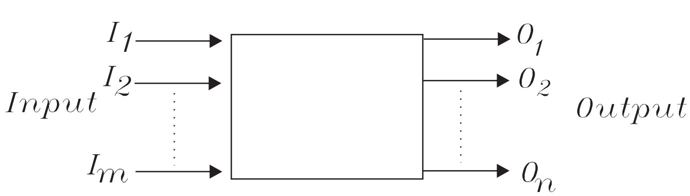

Figure 13.4: Block diagram of a combinational circuit

Combinational circuits are employed in digital systems for generating binary control decisions and for providing digital functions required in data processing. It can be described by a truth table showing the binary relationships between the m input variables and n output variables. There are 2^m possible combinations of binary input values. The truth table lists the corresponding output binary values for each of the 2^m input combinations.

The names, graphic symbols, algebraic functions, and truth tables of eight logic gates are enumerated under the basic circuit elements below. Each gate has one or two binary input variables designated by P and Q and one binary output variable designated by R.

13.7.4 Basic Circuit Elements

Here are the basic circuit elements used in constructing combinational logic circuits:

(i) The AND gate

(ii) The OR gate

(iii) The NOT gate or INVERTER

(iv) The BUFFER

(v) The NAND gate

(vi) The NOR gate

(vii) The exclusive - OR (XOR)

(viii) The exclusive - NOR or equivalence

All these elements may not be used in the same circuit; actually, any logical function can be implemented using just NAND gates, or just NOR gates. However, we shall consider all eight sequentially.

(i) The 'AND' Gate

The AND gate is a circuit which possess a single output line and a number usually between, two and eight, of input lines. It produces the AND logic function: that is, the output is 1 if input P and input Q are both binary 1; otherwise, the output is 0. These conditions are specified in the truth table for the AND gate. The table shows that output R is 1 only when both input P and input Q are 1. The algebraic operation symbol of the AND function is the same as the multiplication symbol of ordinary arithmetic. We can either use a dot between the variables or concatenate the variables without an operation symbol between them. AND gates may have more than two inputs and by definition, the output is 1 if and only if *all* inputs are 1.

For a two-input AND gate shown below, there are only four combinations of 0 and 1 representing F (false) and T (true) respectively as we have in logical propositions:

P	Q	Algebraic function $R = PQ$
1	1	1
1	0	0
0	1	0
0	0	0

Table 13.32: The AND' Gate

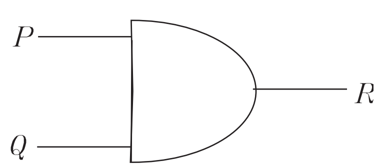

Figure 13.5: Graphical symbol of AND gate

Observe that the output R for the input signals P and Q is given by $R = PQ$. You can draw a three-input AND gate and present its truth table as well.

(ii) The OR Gate - The Inclusive OR

Like the AND gate, an OR gate is a circuit having a single output and several inputs. The OR gate produces the *inclusive - OR* function, that is, the output is 1 if input P or input Q or both inputs are 1; otherwise the output is 0. The algebraic symbol of the OR function is '+', similar to arithmetic addition. OR gates may have more than two inputs and by definition, the output is 1 if any input is 1. Below is a two input OR gate and its truth table:

P	Q	$R = P + Q$
1	1	1
1	0	1
0	1	1
0	0	0

Table 13.33: The OR Gate - The Inclusive OR

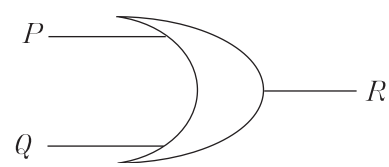

Figure 13.6: Graphical symbol of Inclusive - OR gate

The output R of a two-input OR gate is given as $R = P + Q$, meaning P OR Q.

(iii) The NOT Gate or INVERTER

The NOT gate or INVERTER Circuits inverts the logic sense of a binary signal. It produces the NOT, or *complement*, function. The NOT gate is a very simple circuit which has a single input and a single output. The input signal is simply inverted so that,

Input 0 becomes output 1
Input 1 becomes output 0

This operation is variously known as *inversion negation*, or *complementation*.

The algebraic symbol used for the logic compliment is either a prime or a bar over the variable symbol. However, in this text we shall restrict ourselves to the use of a bar for the compliment or negation of a binary variable. The NOT gate and its truth table is shown below:

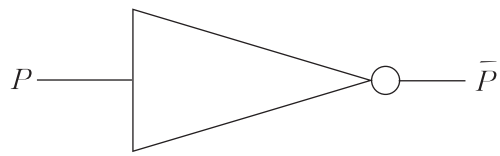

Figure 13.7: Graphical symbol of NOT gate or INVERTER

Input P	Output $\bar{P}$
1	0
0	1

Table 13.34: The NOT Gate or INVERTER

The small circle in the output of the graphic symbol of an inverter and designates a logic compliment.

(iv) Buffer

A *buffer* does not produce any particular logic function since the binary value of the output is the same as the binary value of the input. A triangle symbol by itself designates a buffer circuit. This circuit is used mainly for signal amplification. For example, a buffer that uses $5volts$ for binary 1 will produce an output of $5volts$ when its input is $5volts$. However, the current supplied at the input is much smaller than the current produced at the output. This way, a buffer can drive many other gates requiring a large amount of current not otherwise available from the small amount of current applied to the buffer input. The truth table and the graphic symbol is depicted below:

P	$R = \bar{P}$
O	O
1	1

Table 13.35: Buffer

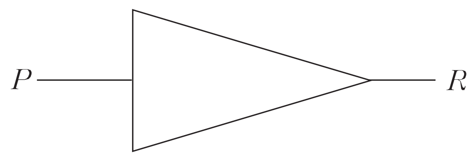

Figure 13.8: Graphical symbol of Buffer

(v) The NAND and the NOR Gates

The NAND function is the compliment of the AND function, as indicated by the graphic symbol which consists of an AND graphic symbol followed by a small circle. The designation, NAND is derived from the abbreviation of NOT-AND. A more proper designation would have been AND-invert, since it is the AND function that is inverted. The NOR gate is the complement of the OR gate and uses an OR graphic symbol followed by a small circle. Both NAND and NOR gates may have more than two inputs, and the output is always the complement of the AND or OR function respectively. The logic symbols for the two - input NAND and NOR gates together with their respective truth tables are shown below:

P	Q	$R = PQ$	$S = (\overline{PQ})$
1	1	1	0
1	0	0	1
0	1	0	1
0	0	0	1

Table 13.36: The NAND Gate

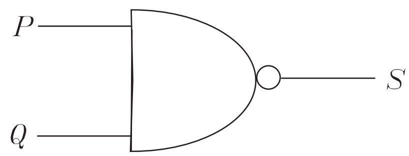

Figure 13.9: Graphical symbol of NAND gate

P	Q	$R = P + Q$	$S = (\overline{P + Q})$
1	1	1	0
1	0	1	0
0	1	1	0
0	0	0	1

Table 13.37: The NOR Gate

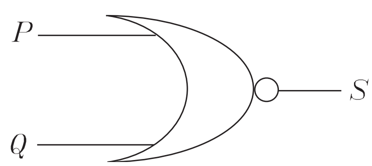

Figure 13.10: Graphical symbol of NOR gate

Observe the NOR gate above have been formed from the inclusive OR, and so is the truth table.

(vi) The Exclusive - OR and Exclusive - NOR Gates

The exclusive - OR gate has a graphic symbol similar to the OR gate except for the additional curved line on the input side. The output of this gate is 1 if any input is 1 but exclude the combination when both inputs are 1. The exclusive - OR function has its own algebraic symbol or can be expressed in terms of AND, OR, and compliment operations as shown in the figure below. The exclusive - NOR is the compliment of the exclusive - OR as indicated by the small circle in the graphic symbol. The output of this gate is 1 only if both inputs have the same binary value; we shall refer to the exclusive - NOR function as the *equivalence function*. Since the exclusive - OR and equivalence functions are not always the compliment of each other, a more fitting name for the exclusive - OR operation would be an *odd function*; i.e. its output is 1 if an odd number of inputs are 1. Thus, in a three-input exclusive - OR (odd) function, the output is 1 if only one input is 1 or if all three inputs are 1. The equivalence function is an *even function*; that is, its output is 1 if an even number of inputs are 0. For a three point equivalence function, the output is 1 if none of the inputs are 0 (all inputs are 1) or if two of its inputs are 0 (one input is 1). Careful investigation will reveal that the exclusive - OR and equivalence functions are the compliment of each other when the gates have an even number of inputs, but the two functions are equal when the number of inputs is odd. These two gates are commonly available with two inputs and only seldom are they found with three or more inputs. The truth tables and the graphic symbols are shown below:-

P	Q	$R = P + Q$
1	1	0
1	0	1
0	1	1
0	0	0

Table 13.38: The Exclusive - OR gate

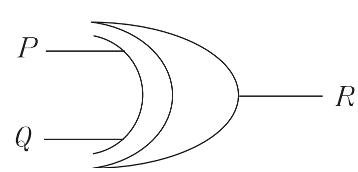

Figure 13.11: Graphical symbol of Exclusive - OR gate

P	Q	$R = PQ$
1	1	1
1	0	0
0	1	0
0	0	1

Table 13.39: Exclusive - NOR gate

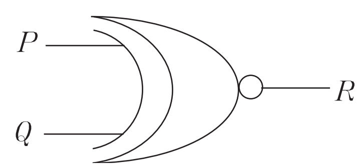

Figure 13.12: Graphical symbol of Exclusive - NOR gate

13.7.5 Duality

A comparison of all the truth tables beginning with that of the AND gate reveals a basic duality which exists between (i) AND gate and OR gates on the one hand and (ii) NAND gates and NOR gates on the other hand.

For AND gate, if the entries in the truth table are changed such that all zeros become ones and all ones become zeros, then the OR gate is obtained while the AND gate is obtained from OR gate by swapping the entries in the truth table as well. This is due to their inherent duality. A similar duality can be established for the NAND and NOR gates using their truth tables. With duality, one can see that a circuit which is an AND gate for positive logic becomes an OR gate when negative logic is used and vice-versa; similarly, for NAND and NOR gates. Usually, the binary values 1 and 0 are represented by voltages in a circuit, such that a system of positive logic is one in which the voltage representing binary 1 is more positive than the voltage representing binary 0. A system of negative logic is correspondingly one in which the voltage representing binary 0 is more positive than that representing binary 1. The concept of duality in which, by the application of positive or negative logic, all ANDs are exchanged for ORs and all ORs are changed to ANDs pervades the theory of switching circuits.

13.7.6 Construction of Logic Circuits

The general procedure for Constructing Logic Circuits are as follows:

1. The problem to be solved is stated and analyzed in terms of truth tables and one or more expressions relating the input and output variable are obtained.

2. These expressions are simplified if possible to obtain minimum circuit.

3. The resultant minimum circuit are finally translated into a network of logic gates which ordinarily, would be used to implement the hardware

Example 13.7.3

Develop the logic expression, $D = \bar{A}.B + A.\bar{B}$ in a form suitable for AND and OR gate manipulation, and hence draw the logic diagram.

Solution

The truth table is as follows:

A	B	$\bar{A}$	$\bar{B}$	$\bar{A}.B$	$A.\bar{B}$	$D = \bar{A}B + A\bar{B}$
1	1	0	0	0	0	0
1	0	0	1	0	1	1
0	1	1	0	1	0	1
0	0	1	1	0	0	0

Table 13.40

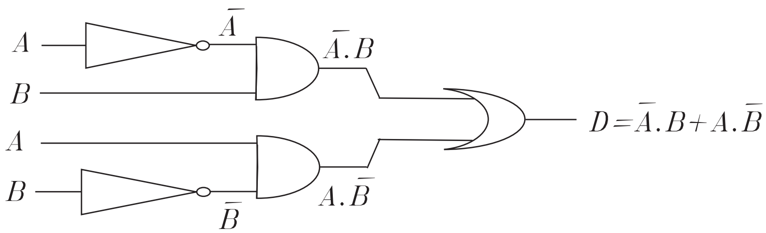

Figure 13.13:

Example 13.7.4

Develop the truth table and hence draw the logic diagram for the expression, $D = A.B.C + \bar{A}.\bar{C}$

Solution

A	B	C	$\bar{A}$	$\bar{C}$	$A.B.C$	$\bar{A}.\bar{C}$	$D = A.B.C + \bar{A}.\bar{C}$
0	0	0	1	1	0	1	1
0	0	1	1	1	0	1	1
0	1	0	1	0	0	0	0
0	1	1	1	0	0	0	0
1	0	0	0	1	0	0	0
1	0	1	0	1	0	0	0
1	1	0	0	0	0	0	0
1	1	1	0	0	1	0	1

Table 13.41

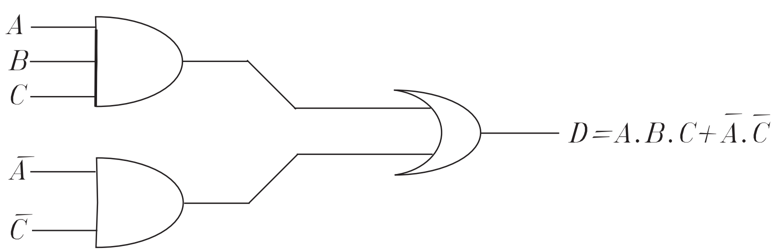

Figure 13.14:

Appendix A

Stirling Formula

Proof. By re-arrangement, we have

$$n!e^n n^{-n-1/2} = \sqrt{2\pi}.$$

Now set

$$a_n = n!e^n n^{-n-1/2} \qquad (i)$$

Except for the factor $\sqrt{2\pi}$, this is exactly the quotient of the LHS and RHS of the formula respectively which we are seeking to prove - and we shall show that it converges.

Applying *ratio test* in equation (i), we have;

$$\begin{aligned}
\frac{a_n}{a_{n+1}} &= \frac{n!}{(n+1)!}\frac{e^n}{e^{n+1}} \cdot \frac{n^{-n-1/2}}{(n+1)^{-n-1-1/2}} \\
&= \frac{n!}{(n+1)n!}e^{-1} \cdot \frac{n^{-n-1/2}}{(n+1)^{-n-1-1/2}} \\
&= \frac{1}{n+1}e^{-1} \cdot \frac{n^{-n-1/2}}{(n+1)^{-n-1-1/2}} \\
&= e^{-1} \cdot \frac{n^{-n-1/2}}{(n+1)^{-n-1-1/2}} \\
&= e^{-1}\frac{(n+1)^{n+1/2}}{n^{n+1/2}} \\
&= e^{-1}\left(\frac{n+1}{n}\right)^{n+\frac{1}{2}} \\
&= e^{-1}\left(\frac{n/n+1/n}{n/n}\right)^{n+\frac{1}{2}} \\
&= \left(1+\frac{1}{n}\right)^{n+\frac{1}{2}} \cdot e^{-1} \\
\therefore \quad \frac{a_n}{a_{n+1}} &= \left(1+\frac{1}{n}\right)^{n+\frac{1}{2}} e^{-1} \qquad (ii)
\end{aligned}$$

Taking the logarithm of both sides, we obtain;

$$\log\left(\frac{a_n}{a_{n+1}}\right) = -1 + (n+\frac{1}{2})\log\left(1+\frac{1}{n}\right)$$

and expanding $\log\left(1+\frac{1}{n}\right)$ into *logarithmic series*, it follows that;

$$\begin{aligned}
\log\left(1+\frac{1}{n}\right) &= \frac{1}{n} - \frac{\left(\frac{1}{n}\right)^2}{2} + \frac{\left(\frac{1}{n}\right)^3}{3} - \dots \\
&= \frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \dots
\end{aligned}$$

Hence,

$$\begin{aligned}
\log\left(\frac{a_n}{a_{n+1}}\right) &= -1+\left(n+\frac{1}{2}\right)\left[\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}\dots\right]\\
&= -1+\left[1-\frac{1}{2n}+\frac{1}{3n^2}\dots\right]+\left[\frac{1}{2n}-\frac{1}{4n^2}+\frac{1}{6n^3}\dots\right]\\
&= \frac{1}{3n^2}-\frac{1}{4n^2}+\frac{1}{6n^3}+\dots
\end{aligned}$$

If we neglect the coefficient of n^3, then;

$$\log\left(\frac{a_n}{a_{n+1}}\right)=\frac{1}{12n^2}+\dots \qquad (iii)$$

For large values of n the first term predominates, and so

$$\log\left(\frac{a_n}{a_{n+1}}\right)>0$$

Thus, taking the exponential of both sides, we obtain;

$$\frac{a_n}{a_{n+1}}>e^\circ$$

$$\frac{a_n}{a_{n+1}}>1 \qquad (iv)$$

$$a_n>a_{n+1} \qquad (v)$$

On the long run, the a_n form a *decreasing sequence* of positive numbers; and have a limit p.

So for $p>0$, if we put;

$$b_n=\log a_n-\frac{1}{11(n-1)} \qquad (vi)$$

then after a while,

$$\begin{aligned}
b_n-b_{n+1} &= \log a_n-\frac{1}{11(n-1)}-\log a_{n+1}+\frac{1}{11(n+1-1)}\\
&= \log\left(\frac{a_n}{a_{n+1}}\right)-\left[\frac{1}{11(n-1)}-\frac{1}{11n}\right]\\
&= \log\left(\frac{a_n}{a_{n+1}}\right)-\left[\frac{n-n+1}{11n(n-1)}\right]\\
\therefore \quad b_n-b_{n+1} &= \log\left(\frac{a_n}{a_{n+1}}\right)-\frac{1}{11n(n-1)}\dots \qquad (vii)
\end{aligned}$$

Now substituting for (iii) in (vii), we obtain;

$$\begin{aligned} b_n - b_{n+1} &= \log\left(\frac{a_n}{a_{n+1}}\right) - \frac{1}{11n(n-1)} \\ &= \frac{1}{12n^2} - \frac{1}{11n(n-1)} \dots \qquad (viii) \end{aligned}$$

which becomes negative, i.e., from a certain k onwards, such that,

$$b_k < b_{k+1} < \ldots < b_n$$

Therefore,

$$\log a_n > b_k$$

It follows that,

$$p = \lim a_n \geq e^{-b_k} > 0$$

We assume here that $p = \sqrt{2\pi}$, which therefore concludes the proof.
□

Bibliography

[1] A.P. Armit, *Advanced Level Vectors,* Heimann Educational Books, London, 1973.

[2] Alden T. Willis and C.L. Johnston, *Intermediate Algebra,* Wadsworth Publishing Co. California, 1981.

[3] Allen Walker Read et al, *Standard Dictionary of English Language,* International Edition, Funk and Wagnalls Company, New York, 1959.

[4] B.P. Demidovich and I.A. Maron, *Computational Mathematics,* MIR Publishers, Moscow, 1973.

[5] B.R. Thakur et al, *Vector Analysis,* Ram Prasad and Sons: ARRA-3, 1969.

[6] C.E. Weatherburn, *Elementary Vector Analysis (with Applications to Geometry and Mechanics),* G. Bell and Sons Ltd., 1967.

[7] Charles F. Brumfiel et al, *Fundamental Concepts of Elementary Mathematics,* Addison-Wesley Publishing Co. INC., London, 1962.

[8] David Holland and T. Treeby, *Vectors - Pure and Applied,* Edward Arnold (Publishers) Ltd., London, 1979.

[9] E.H. Thompson, *An Introduction to Algebra of Matrices with some Applications,* Adam Hilger, London, 1969.

[10] Elmer B. Mode, *Elements of Statistics,* Prentice-Hall Mathematic Series, 1971.

[11] G.N. Yakovlev, *Higher Mathematics,* MIR Publishers, Moscow, 1987.

[12] Garret Birkoff and Saunders Maclane, *A Survey of Modern Algebra,* The Macmillan Company, New York, 1950.

[13] Hans Freudenhal, *Probability and Statistics,* Elsevier Publishing Company, N.Y., 1965.

[14] J.F. Albert, A. Godman and G.E.O. Ogun, *Additional Mathematics for West Africa,* Longman Group Ltd., Essex, 1984.

[15] Joseph Blakey, *Intermediate Pure Mathematics,* The MacMillan Press Ltd., London, 1977.

[16] M. Morris Mano, *Computer System and Architecture,* Prentice-Hall of India, New Delhi, 1990.

[17] M.K. Potapov et al, *Algebra and Analysis of Elementary Functions,* MIR Publishers, Moscow, 1987.

[18] Michael Swan, *Practical English Usage,* Low–Price Edition, Oxford University Press, London, 1975.

[19] Ralph Crouch, *Finite Mathematics with Statistics for Business,* McGraw-Hill Book Company, New York, 1968.

[20] Richard A. Brualdi, *Introductory Combinatorics,* New York, 1977.

[21] S.A. Ilori and O. Akinyele, *Elementary Abstract and Linear Algebra,* Ibadan University Press, Ibadan, 1986.

[22] S.U. Egarievwe and S.C. Chiemeke et al, *General Introduction to Computer Science,* Anchor Educational Press, Lagos, 1992.

[23] Seymour Lipschutz, *Schaum's Outline of Theory and Problem of Essential Computer Mathematics,* Schaum's Outline Series, McGraw-Hill Book Company, New York, 1982.

[24] W. Gilbert, S. Gollwald, M. Hellwich, H. Kastner and H. Kustner, *The VNR Concise Encyclopedia of Mathematics,* Van Nostrand Reinhold, New York, 1989.

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