INTRODUCTION TO REAL ANALYSIS

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PART 1: SETS

Unit 1: Cartesian Products

1.1 Introduction

The idea of a set is basic to all mathematics and mathematical objects, and constructions are based on set theory. The knowledge of set theory helps us to classify items/elements into groups, subgroups, family, or classes according to a certain common attributes.

It is assumed that students offering this course must have completed an elementary course in basic set theory at their 100L; we shall therefore not discuss basic set operations but shall move a step further to discuss in this module other topics in set theory that include Cartesian Products, Function and Mapping as well as Countable sets. We shall denote a set by an upper case letter such as A,B,X,Y, etc., and elements by lower case letters such as a,b,x,y, etc. We shall specify a set in any two forms – the first is sometimes referred to as the *roster method* in which the objects of a set is listed in braces separated by commas such as

$$\{a, e, i, o, u\}$$

which describes the set of all vowels in the English alphabet. Alternatively, we can describe some property held by all elements which we shall refer to as the *property method* and is denoted by

$$\{x : x \text{ is a vowel}\}$$

read as "the set of all elements x such that x is a vowel".

We shall often have instances when to combine together the sets of a given class into a single new set called their product. In this unit, we shall discuss the significance and properties of this product relationship between two sets, say A and B. This product concept was heralded by the (x, y) coordinate plane of analytic geometry – a plane equipped with the usual rectangular coordinate system.

1.2 Objective

The essence of the Cartesian Product is to enable us to use algebraic tools in geometric arguments, for example, the algebraic interpretation for the familiar rectangular coordinate plane. It equally lays the foundation for the study of topological spaces such as the Euclidean space. At the end of this unit, we should be familiar with the notation R, the real line as in R^1, R^2 as in coordinate plane which consists of the X and Y axis; and whenever the index, say n of R is greater than 2 i.e., when $i = 3, 4, 5, ..., n$, which leads us to n-dimensional coordinate space referred to as the Euclidean Space.

1.3 Ordered Pairs

In order for us to progress evenly to the discussion on Cartesian Product, it will be necessary to first consider Ordered Pairs – which forms the basis for product sets. Suppose we have two elements a, b, such that the element a is designated as the first element; and the element b as the second element, then an ordered pairs of the two elements is written as (a, b); with the understanding that only two pairs (a, b) and (c, d) are equal if and only if

$$a = c \text{ and } b = d.$$

1.4 Product Sets

Let A and B be two arbitrary sets, then the *Product Set* of A and B can be written as $A \times B$ which consists of all ordered pairs (a, b), where $a \in A$ and $b \in B$; i.e.,

$$A \times B = \{(a, b) : a \in A, b \in B\}$$

Generally, the product of a set with itself, say $A \times A$, can be denoted by A^2. For example, let $X = (1, 2, 3)$ and $Y = (y_1, y_2)$; then

$$X \times Y = \{(1, y_1), (1, y_2), (2, y_1), (2, y_2), (3, y_1), (3, y_2)\}$$

The concept of product set can be generalized to any finite number of sets in a natural way. Thus the product sets of $P_1, ..., P_n$, is denoted by

$$P_1 \times P_2 \times ... \times P_n, \text{ or } \prod_{i=1}^{n} P_i$$

which consists of n-tuples $(p_1, p_2, ..., p_n)$, where $p_i \in P_i$ for each i. The Cartesian product is an off-shoot of product sets, even in most books, both product set and Cartesian sets are used synonymously.

Definition 1 *Let A and B be any two non-empty sets with $a \in A$ and $b \in B$, then the Cartesian Product given by $A \times B$ of A and B is the set of all ordered pairs (a, b).*

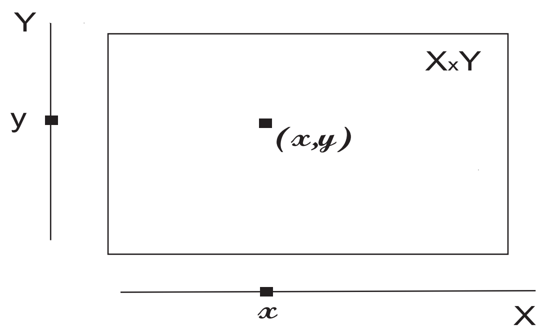


Figure 1.1: A way of visualizing $X \times Y$.

In spite of the arbitrary nature of X and Y, their product can be represented in the Figure 1.1 which is loosely similar to the usual coordinate plane. The term, product is applied to this set and it is thought of as the result of "multiplying together" X and Y such that if X and Y are finite sets with m and n elements, then clearly, $X \times Y$ has mn elements.

1.5 The Coordinate Plane

The Cartesian product of two sets can be easily extended to the case of n sets for any positive integer n. Thus if $R_1 \times R_2 \times ... \times R_n$ is the set of all ordered n-tuples $(r_1, r_2, ..., r_n)$, where $r_i \in R_i$ for each i.

If the R's are all replicas of a single set R such that

$$R_1 = R_2 = ... = R_n = R,$$

then their product can be denoted by R^n. Thus, R^1 is just R – the real line, i.e.,

and R^2 is the coordinate plane, i.e.,

Similarly, R^3 is the set of all ordered tripples of real numbers – underlies solid analytic geometry which arises through the introduction of a rectangular coordinate system into ordinary three-dimensional space.

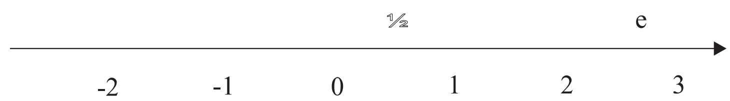


Figure 1.2: The real line.

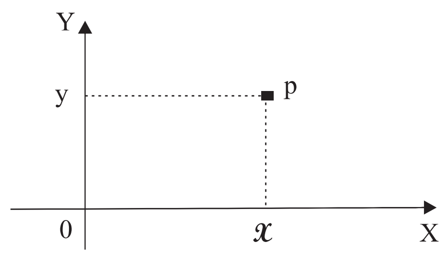
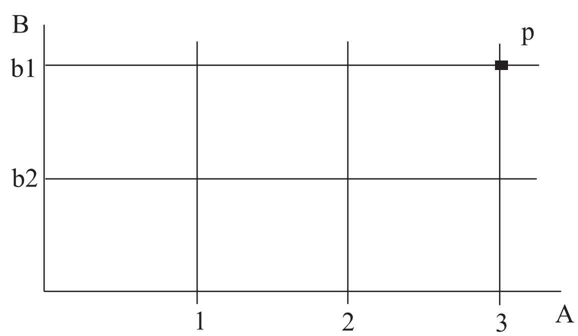


Figure 1.3: The coordinate plane.

Examples 1.1

1. The Cartesian plane $R^2 = R \times R$, as given in Figure 1.3 above. The point P represents an ordered pair (x, y) of real numbers.

2. Let $A = \{1, 2, 3\}$ and $B = \{b_1, b_2\}$.



Then

$$A \times B = \{(1, b_1), (1, b_2), (2, b_1), (2, b_2), (3, b_1), (3, b_2)\}$$

Observe that the above vertical lines through the points of A and the horizontal lines through the points of B meet in 6 points which represents $A \times B$. The point P is ordered pair $(3, b_2)$.

1.6 Conclusion

It is assumed that we have covered the fundamental idea of Cartesian product with a view to paving the way for discussion on product sets in general, which you may encounter throughout this course and in topological space. We shall discuss functions and mappings in our next unit.

1.7 Summary

Our main discuss in this unit is Cartesian product and have provided the relevant background ideas such ordered pairs, product sets that provided the ingredients for the proper understanding of the fundamental concept required. We concluded by providing relevant examples.

1.8 Tutor Marked Assignments

1. Let R be the relation $<$ from $A = \{1, 2, 3, 4\}$ to $B = \{1, 3, 5\}$, i.e., $(a, b) \in R$ if and only if $a < b$:
   (a) Write R as a set of ordered pairs.
   (b) Plot R on a coordinate diagram of $A \times B$.
2. Let S be the set of all real numbers whose squares are equal to 25. Show how to describe S by:
   (a) The roster method
   (b) The property method.

1.9 References

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Unit 2: Functions and Mappings

1.1 Introduction

Literally, a function could be said to be something closely related to or dependent on something else. But in mathematics, it will be seen that a function is a special kind of a set, which are often helpful and suggestive about an idea. Ordinarily, the word *function* to a student of elementary mathematics meant a definite formula such as

$$f(x) = x^2 - 4$$

which associates to each real number x to another real number $f(x)$. The function $f(x)$ as mentioned above is defined for all real numbers of x. However, certain formulas such as

$$g(x) = \sqrt{x - 3}$$

do not give rise to real numbers for all real values of x. Equally, the formula

$$y = \log x$$

is defined only for positive values of x. Similarly, the absolute value of a real number,

$$h(x) = |x|$$

is an honest function but the definition is given in "pieces" by

$$|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$

It is however increasingly explicit that the requirement that a function be a formula was unduly restrictive and that a more general definition would be useful. It is also evident that it is important to make a clear distinction between the function itself and the values of the function.

1.2 Objective

Most real life problems in the engineering and sciences are often represented in the form of functions. The knowledge of the concept of functions and mappings is very significant in the visualization of most mathematics problems and studies in mathematical analysis such as in differentiation and integral theory. At the end of this lecture, we should be able to understand the real concept of function as different from ordinary formula to a relationship that exists between variables or elements in a system and how such can be visualized or represented in mathematics.

1.3 Graphical Representation

Suppose X and Y are two sets, then a "*function*" from X and Y is a set f of ordered pairs $X \times Y$ with the property that if (x, y) and (x, y') are elements of f, then $y = y'$. The set of all elements of X that can occur as first elements of f is called the *Domain* of f and will be denoted by $D(f)$. (see figure 1.4).

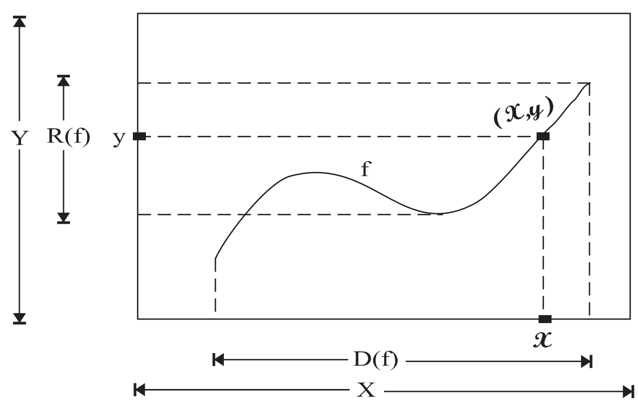

Figure 1.4: A function as a graph.

Similarly, the set of all elements of Y that can occur as second members of elements of f is called *Range* of f (or the set of values of f) and will be denoted by $R(f)$. In other words, a function consists of three objects: two non-empty sets X and Y (which may be distinct or equal), and a rule – which assigns to each element x in X a single fully determined element y in Y. The y which corresponds in this way to a given x is usually written as

$$y = f(x)$$

instead of $(x, y) \in f$, and we often refer to the element b as the value of f at the point x. Sometimes, we say that y is the image under f of the point x. This notation is supposed to be suggestive of the idea that the rule f takes the element x and does something to it to produce the element $y = f(x)$. The rule f is often called a *mapping* or *transformation*, or *operator*. Thus, we say that f is a mapping of x's to y's, or transforming x's into y's, or operating on x's to produce y's, and we refer to x as the *independent* variable while y is the *dependent* variable.

We often denote by $f : X \mapsto Y$ the function with rule f, domain X, and range contained in Y or we say that f is a mapping of X to Y.

1.4 Tabular Representation

One way of visualizing a function is as a graph as contained in Figure 1.4 above. Another way which is equally important and widely used is as table. Consider Table 2.1 which consists of the exam scores of student in mathematics:

Student Name	Scores
Andrew	52
Wale	58
Kazim	65
Kome	72
Lawal	48

Observe that the domain of this score's function f consists of 5 students:

$$D(f) = \{\text{Andrew,Wale,Kazim,Kome,Lawal}\}$$

while the range of the function consists of five scores,

$$R(f) = \{52, 58, 65, 72, 48\}$$

Thus, the actual elements of the function are ordered pairs

$$\text{(Andrew,52),(Wale,58),(Kazim,65),(Kome,72),(Lawal,48)}$$

From the above table, we could say that the value (score) of this exam-score function f at Andrew is 52, and thus write $f(\text{Andrew}) = 52$, and so on. It is assumed that we are familiar with such use of tables to convey information. They are important examples of functions and are usually of a nature that would be difficult to express in terms of a formula.

1.5 Transformations

Another way of visualizing a function is as a transformation of part of a set A into part of another set B. In this analogy, when $(a, b) \in f$, we think of f as taking the element a from the subset $D(f)$ of A and "transforming" or "mapping" it into an element $b = f(a)$ in the subset $R(f)$ of B. This is illustrated by the diagram such as Figure 1.5 below:

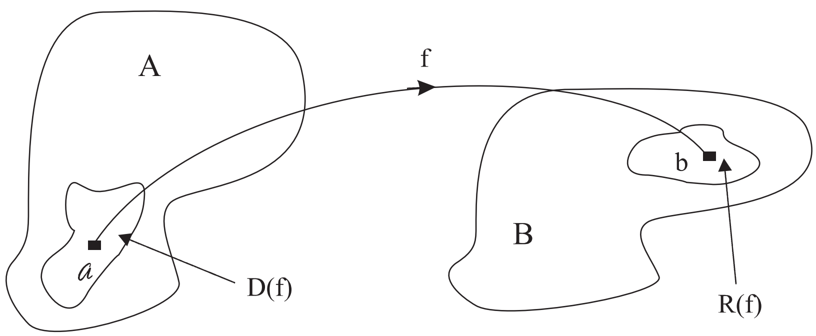

Figure 1.5: A function as a transformation.

Similarly, another way of visualizing a function is as a machine which accepts elements of $D(f)$ as inputs and produce a corresponding elements of $R(f)$ as outputs:

Figure 1.6: A function as a machine.

Observe that if we take an element x from $D(f)$ and pass it into f, then it produces a corresponding value $f(x)$. If we pass a different element, say y of $D(f)$ into f, we get $f(y)$ (which may or may not differ from $f(x)$). However, if we try pass something different which does not belong to $D(f)$ into f, we may find out that it is not accepted, for f can operate only on elements belonging to $D(f)$.

Example 1.1

Let $f : R \mapsto R$ be the function which assigns to each real number its square, i.e., for each $x \in R$, $f(x) = x^2$. Here, f is a real–valued function. Its graph, $\{(x, x^2) : x \in R\}$, is displayed in Figure 1.7 below:

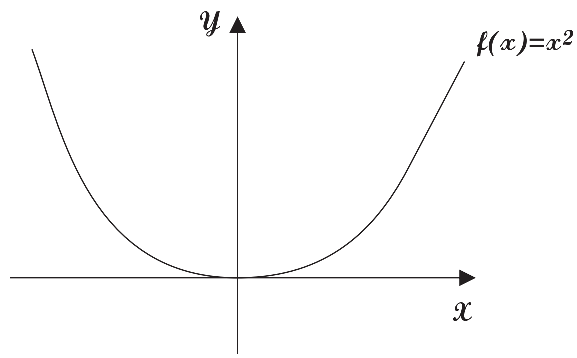

Figure 1.7: The graph of $f(x) = x^2$.

The range of f is the set of non–negative real numbers, i.e., $f(R) = \{x : x \in R, x \geq 0\}$.

1.6 Direct and Inverse Images

1.6.1 Direct Image

Let f be an arbitrary function with domain $D(f)$ in A and range $R(f)$ in B. We do not assume that f is one-to-one.

Definition 2 *If E is a subset of a A, then the* direct image *of E under f is the subset $R(f)$ given by*

$$\{f(x) : x \in E \cap D(f)\}$$

For clarity, we sometimes denote the direct image of a set E under f by the notation $f(E)$. (see figure 1.8 below).

It will be observed that if $E \cap D(f) = \emptyset$, then $f(E) = \emptyset$. However, if E consists of a single point p in $D(f)$, then the set $f(E)$ consists of the single point $f(p)$. Certain properties of sets are preserved under the direct image — we shall state these properties without proof.

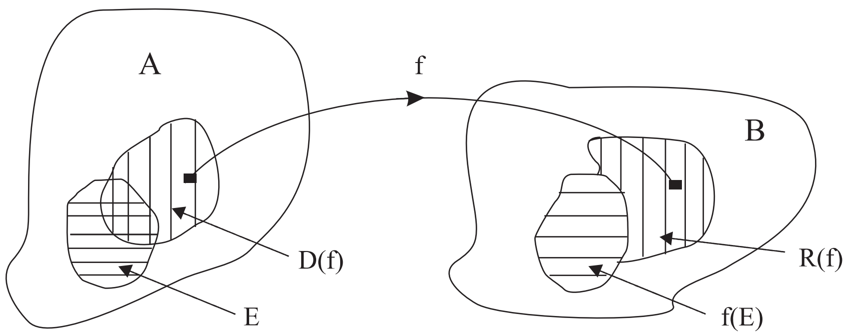

Figure 1.8: Direct images.

Let f be a function with domain A and range in B, and let E, F be subsets of A, then the following conditions are satisfied:

1. If $E \subseteq F$, then $f(E) \subseteq f(F)$,
2. $f(E \cap F) \subseteq f(E) \cap f(F)$,
3. $f(E \cup F) \subseteq f(E) \cup f(F)$,
4. $f(E \setminus F) \subseteq f(E)$.

We will prove 1 and 2, while the proof of 3 and 4 will be left as exercise.

Proof.

1. If $x \in E$, then $x \in F$ and hence $f(x) \in f(F)$. Since this is true for all $x \in E$, we infer that $f(E) \subseteq f(F)$.
2. Since $E \cap F \subseteq E$, it follows from (1) above that $f(E \cap F) \subseteq f(E)$.

Similarly,

$$f(E \cap F) \subseteq f(F)$$

Therefore, we conclude that

$$f(E \cap F) \subseteq f(E) \cap f(F).$$

1.6.2 Inverse Image

We now introduce the notion of the inverse image of a set under a function. Note that it is not required that the function be one–to–one.

Definition 3 *If H is a subset of B, then the* inverse image *of G under f is the subset of $D(f)$ given by*

$$\{x : f(x) \in G\}$$

We can denote the inverse image of a set H under f by the symbol $f^1(G)$. (see Figure 1.9 below)

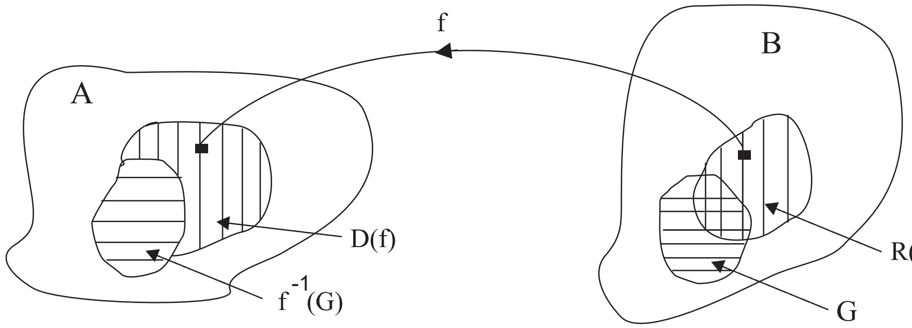

Figure 1.9: Inverse image.

Once again, we emphasize that f need not be one–to–one so that inverse function f^{-1} need not exist. However, if f^{-1} does exist, the $f^{-1}(G)$ has a direct image of G under f^{-1}. We state the following set properties of inverse image.

Let f be a function with domain A and range in B, and let G, H be subsets of B, then the following conditions are satisfied:

1. If $G \subseteq H$, then $f^{-1}(G) \subseteq f^{-1}(H)$,

2. $f^{-1}(G \cap H) = f^{-1}(G) \cap f^{-1}(H)$,

3. $f^{-1}(G \cup H) = f^{-1}(G) \cup f^{-1}(H)$,

4. $f^{-1}(G \setminus H) = f^{-1}(G) \setminus f^{-1}(H)$.

We shall prove 1 and 3, while 2 and 4 shall be left as exercise.

Proof.

1. Suppose that $x \in f^{-1}(G)$; then by definition $f(x) \in G \subseteq H$. Hence, $x \in f^{-1}(H)$ and $f^{-1}(G) \subseteq (H)$.

3. Since G and H are subsets of $G \cup H$, it follows from 1 above that

$$f^{-1}(G \cup H) \supseteq f^{-1}(G) \cup f^{-1}(H).$$

Conversely, if $x \in f^{-1}(G \cup H)$, then $f(x) \in G \cup H$. It follows that either $f(x) \in G$, which implies that $x \in f^{-1}(G)$, or $f(x) \in H$, in which case $x \in f^{-1}(H)$.

Hence,

$$f^{-1}(G \cup H) \supseteq f^{-1}(G) \cup f^{-1}(H).$$

1.7 Conclusion

We have observed from our discussion that many kinds of functions occur in mathematics in a great variety of situations. Contrary to elementary meaning of functions, we have discussed the general concept of a function in a more broader and deeper detail to encompass its abstract properties. We shall be dealing with Countable Sets in our next unit.

1.8 Summary

In this unit, we have covered the general concept of a function using graphical illustrations and tabular as well as machine visualizations. We equally discussed some of the abstract properties of a function such as direct and inverse images.

1.9 Tutor Marked Assignments

1. Consider the functions

$$f = \{(1,3),(2,5),(3,3),(4,1),(5,2)\}$$

$$g = \{(1,4),(2,1),(3,1),(4,2),(5,3)\}$$

from $X = \{1,2,3,4,5\}$ into X. Hence determine the range of f and g.

2. Consider the functions, $f = \{(1,4),(2,1),(3,4),(4,2),(5,4)\}$ from $A = \{1,2,3,4,5\}$ such that $f : A \mapsto A$. Find (a) $f[\{1,3,5\}]$, (b) $f^{-1}[\{3,5\}]$.

3. Consider the function $f : R \mapsto R$ defined by $f(x) = x^2$. Find (a) $f^{-1}[\{25\}]$, (b) $f^{-1}[\{-9\}]$, (c) $f^{-1}[\{x : x \leq 0\}]$.

1.10 References

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Unit 3: Countable Sets

1.1 Introduction

The subject of this unit though elementary, lies at the foundation of modern mathematics. It is a vital instrument in the day-to-day work of many mathematicians, and we shall make extensive use of it in subsequent topics in this course. This story which was created by the German mathematician, *Cantor*, also has great anesthetic appeal for it begins with ideas of extreme simplicity and develops it through natural stages into an elaborate and beautiful structure of thought. In the course of our discussion, we shall cover such elementary topics such as cardinal numbers, natural numbers as well as finite and infinite sets. Our coverage also include supremum and infimum of real numbers.

1.2 Objective

At the end of this lecture we should be able to relate our elementary knowledge of counting with the concept of countable sets and differentiate between what constitute a countable set and uncountable set. How are they related to finite and infinite sets which we shall be discussing at the last module of this course? Finally, we shall be discussing supremum and infimum which in ordinary sense means maximum and minimum, their notation and how they are used in mathematical analysis.

1.3 Cardinal and Natural Numbers

Cardinal numbers are used in counting,such as the positive integers (or natural numbers) 1,2,3,... that we are familiar with from elementary school. But there is much more to the story than what we are already familiar with. The set of counting is undoubtedly one of the oldest of human activities. Men probably learn to count in a crude way at about the same time as they began to develop articulate speech. The earliest men who lived in communities and domesticated animals must have found it necessary to record the number of goats in the village by means of a pile of stones and some similar devices. The simple and yet profound idea of a one-to-one correspondence between the stones and goats would have fully met the needs of the situation.

In a manner of speaking, we ourselves use the set of all positive integers,

$$N = \{1, 2, 3, ...\}$$

as a “pile of stones”. We carry this set around with us as part of our intellectual equipment. However, our procedure is slightly more sophisticated than that of the primitive savage. We have the symbols 1,2,3,... for the numbers which arise in counting; we can record them for future use, and communicate them to other people, and manipulate them by the operations of arithmetic. But the underlying idea, that of the one-to-one correspondence, remains the same for us as it probably was for early men.

1.4 Finite and Infinite Sets

The positive integers are adequate for the purpose of counting any non-empty finite set, and since outside of mathematics, all sets appear to be of this kind, they suffice for all non-mathematics counting. But in the world of mathematics, we are obliged to consider many infinite sets, such as the set of all positive integers itself, the set of all integers, the set of all rational numbers, the set of all real numbers, the set of all points in a plane, and so on. It is often important to be able to count such sets, and it was Cantor’s idea to do this, and to develop a theory of infinite cardinal numbers, by means of a one-to-one correspondences.

In comparing the sizes of two sets, the basic concept is that of numerical equivalence of the ordered pairs. Two non-empty sets A and B are said to be numerically equivalent if there exists a one-to-one mapping of one onto the other, or – and this amounts to the same thing – if there can be found a one-to-one correspondence between them. To say that two non-empty finite sets are numerically equivalent is of course to say that they have the same number of elements in the ordinary sense. If we count one of them, we simply establish a one-to-one correspondence between its elements and a set of positive integers of the form $\{1, 2, 3, ..., n\}$, and we then say that n is the number of elements possessed by both, or the cardinal number of both. The positive integers are the finite cardinal numbers.

Thus all that is necessary to show that an infinite set X is numerically equivalent to the set of natural number N, is that we are able list the elements of X, with a first, a second, a third and so on in such a way that it is completely exhausted by this counting off of its elements. It is for this reason that any infinite set which is numerically equivalent to N is said to be *countably infinite*. Thus, we say that a set is *countable*, if it is non-empty and finite; in which case it can be obviously counted or it is countably infinite or uncountable. Note that Cantor discovered that the infinite set R of all real numbers is not countable – or we may say R is uncountable or uncountably infinite. Since we customarily identify the element of R with the points of the real line, this amounts to the assertion that the set of all points on the real line represents a higher type of infinity than that of only the integral points or only the rational points. For example, the set I of real numbers x satisfying $0 \leq x \leq 1$ is not countable. Cleanly, there are infinitely many rational numbers in the interval $0 \leq x \leq 1$, thus the set I cannot be finite.

1.5 Suprema and Infima

We now introduce the concept of an upper bound and lower bound of a set of real numbers. This knowledge will be of utmost importance in later part of this course.

Definition 4 *Let E be the subset of R. An element u of R is said to be an upper bound of E if $e \leq u$ for $e \in E$. Similarly, an element l of R is said to be a lower bound of E if $l \leq e$ for all $e \in E$.*

It should be observed that a subset E of R may not have an upper bound; but if it has one, then it has infinitely many. For example, if

$$E_1 = \{x \in R : x \geq 0\},$$

the E_1 has no upper bound. Similarly, the set $E_2 = \{1, 2, 3, ...\}$ has no upper bound. The situation is not different for the interval

$$E_3 = \{x \in R : 0 < x < 1\}$$

which has 1 as an upper bound, in fact, any real number $u \geq 1$ is also an upper bound of E_2. Again, the set of

$$E_4 = \{x \in R : 0 \leq x \leq 1\}$$

has the same upper bound as E_3. However, the student may note that E_4 actually contains one of its upper bounds. Note that any real number is an upper bound for the empty set.

When a set has an upper bound, we shall say it is *bounded above*; and when a set has a lower bound, then it is *bounded below.* If E is bounded both below and above, we may say that it is *bounded.* While if E is not having either a lower bound or upper bound, then it is *unbounded*. For example, both E_1 and E_2 above are unbounded but are bounded below, whereas both E_3 and E_4 are bounded.

Definition 5 *Let E be a subset of R which is bounded above. An upper bound of E is said to be a* supremum *(or a* least upper bound*) of E if it is less than any upper bound of E. Similarly, if E is bounded below, then a lower bound of E is said to be* infimum *(or* greatest lower bound*) of E if it is greater than any other lower bound E.*

Alternatively, a real number u is said to be supremum of a subset E if it satisfies the following conditions:

1. $e \leq u \ \ \forall \ e \in E$;

2. if $e \leq v \ \ \forall \ e \in E$, then $u \leq v$.

The first condition makes u an upper bound of E, and the second makes it less than or equal to, any upper bound of E.

Note that there can be only one supremum for a given set. For instance, suppose $u_1 \neq u_2$ are both supremum of E; then they are both upper bound of E. Since u_1 is a supremum of E and u_1 is an upper bound of E, we must have $u_1 \leq u_2$. A similar argument gives $u_2 \leq u_1$, showing that $u_1 = u_2$, which is a contradiction. Hence a set E can have at most one supremum and a similar argument shows that it can have at most one infimum. When these numbers exist, we sometime denote them by $supE$ or $infE$.

1.6 Conclusion

We have found out that the necessary condition for a set to be countable is that the set must be non-empty and finite, otherwise it is uncountable or infinitely countable. Whereas a set is said to be finite if it has a one-to-one correspondence with the set N of natural numbers, $\{1, 2, 3, ...\}$. This unit marks the end of module 1 in this course. Module 2 will be concerned with limits and sequences of real numbers.

1.7 Summary

We introduced the work of Cantor as the major contributor to the subject of countable set through his work on Cantor set. Our areas of coverage include cardinal or natural numbers, finite and infinite sets and finally supremum and infimum.

1.8 Tutor Marked Assignments

1. Give two examples each of a countable and non-countable sets, and hence justify your choice.
2. Prove that every subset of countable set is countable.
3. If Q denotes the set of rational numbers, prove that the following set is countable: $\{x : x \in Q, x \geq 1\}$.
4. Prove that the set of real numbers in [0,1] is countably infinite.

1.9 References

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PART 2: LIMITS AND SEQUENCES OF REAL NUMBERS

Unit 1: Limits and Convergence of Sequences

1.1 Introduction

The mathematical concept of limit is a particularly difficult notion, typical of the kind of thought required in advanced mathematics. It hold a central position which permeates the whole of mathematical analysis – as a foundation of the theory of approximation, of continuity, and of differential and integral calculus.

One of the greatest difficulties in teaching and learning the limit concept lies not only on its richness and complexity, but also in the extent to which the cognitive aspects cannot be generated purely from mathematical definition. The distinction between the definition and the concept itself is remarkably very important. Remembering the definition of a limit is one thing, acquiring the fundamental conception is another. One fact is the idea of approximation, usually first encountered through a dynamic notion of limit, and the way in which the concept of limit is put to work to resolve real problems which rely not on the definition but on many diverse properties of the intuitive concept. Starting from such point of view, students often believe that they "understand" the definition of a limit without truly acquiring all the implications of formal concept. Student are often able to complete many of the exercises they are asked to perform without having to understand the formalism of the definition at all. Meanwhile, the quantifiers "for all", "there exists", which occur in epsilon-delta definitions have their own meanings in everyday language subtly different from those encountered in the definition of limit concept. From such beginning arise conceptual obstacles which may cause some difficulties in comprehension; we shall therefore introduce this unit with examples as much as possible in order to dilute the perceived difficulties in your understanding. We shall be discussing the elementary properties of limit, while in unit 2, we shall continue with continuity of convergence of sequences.

1.2 Objective

The concept of limit is required to provide the basis for the kind of thought necessary for advanced mathematical analysis – as a foundation for the theory of approximation, of continuity, and differential and integral calculus. At the end of the lecture we should be able to understand when a sequence of real numbers converges, obtain the limit or point of convergence, or establish non-existence of convergence.

1.3 Sequences in R

In order to understand the concept of limits, it is necessary that we review the notion of sequences in the real line as we have already known in elementary mathematics.

Definition 6 *A sequence in R^p is a function whose domain is the set of natural numbers, $N = \{1, 2, 3, ...\}$, and whose range is contained in R^q.*

In defining sequences, we often list in order the elements of the sequence, stopping when the rule of formation seems evident. Thus, we write

$$\{2, 4, 6, 8, ...\}$$

for the sequence of even integers. A more satisfactory method is to specify a formula for the general term of the sequence, such as

$$\{2n : n \in N\}.$$

In practice, it is often more convenient to specify the value x_1 and a method of obtaining

$$x_{n+1}, \ \ n \geq 1,$$

when x_n is known. However, many methods of defining a sequence are possible.

Example 4.1

1. $x_n = \frac{1}{n}$. The sequence is $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ...$

2. $x_n = n^2$. The sequence is 1,4,9,16,...

3. $x_n = \frac{(-1)^n}{n}$. The sequence is $-1, \frac{1}{2}, \frac{-1}{3}, \frac{1}{4}, ...$

1.4 Elementary Properties of Sequences

Let $X = (x_n)$ and $Y = (y_n)$ and sequences in R, then the following condition holds:

1. $X + Y = (x_n + y_n)$.

2. $X - Y = (x_n - y_n)$.

3. $XY = (x_n y_n)$.

4. $X/Y = (x_n/y_n)$, provided $y \neq 0$.

For example, if X, Y are the sequences in R given by

$$X = (2, 4, 6, ..., 2n, ...), \quad Y = (1, \frac{1}{2}, \frac{1}{3}, ..., \frac{1}{n}, ...)$$

then we have

$$X + Y = (3, \tfrac{9}{2}, \tfrac{19}{3}, \ldots, \tfrac{2n^2+1}{n}, \ldots)$$

$$X - Y = (1, \tfrac{7}{2}, \tfrac{17}{3}, \ldots, \tfrac{2n^2-1}{n}, \ldots)$$

$$XY = (2, 2, 2, \ldots, 2, \ldots)$$

$$\tfrac{X}{Y} = (2, 8, 18, \ldots, 2n^2 \ldots)$$

We now come to the notion of the limit of a sequence.

1.5 Limit of a Sequence in R

Let $X = (x_n)$ be a sequence in R. An element x of R is said to be a limit of X if for each positive real number $\epsilon > 0$, there exists a positive integer n_0 such that if $n > n_0$ then,

$$|x_n - x| < \epsilon.$$

If a limit exists then the sequence converges and it is unique. The limit or convergence is denoted by

$$\lim_{n\to\infty} x_n \to x$$

Simple Examples of Limits

Let us consider some examples of sequences and try to see what their limits are:

1. The simplest sequence is a constant sequence which keeps repeating itself. For instance

$$x_n = 5, 5, 5, ...$$

   The limit of this sequence as $n \to \infty$ is clearly 5, since the sequence is stuck at 5.

2. The sequence

$$x_n = -1, 4, 5, 7, 8, 8, 8, ...$$

   which eventually stabilizes at the constant value 8 has limit of 8 as $n \to \infty$.

3. In contrast, the sequence

$$x_n = 1, 3, 4, 1, 3, 4, 1, 3, 4, ...$$

   does not have a limit. For example, the point 3 cannot be the limit of the sequence because for instance, (2.5,3.5), is within the neighborhood of 3, but the sequence keeps falling outside this neighborhood (when its limits is 1 or 4).

4. Let $X = (x_n)$ be a sequence in R defined by

$$x_n = \frac{2\mathrm{n}+1}{\mathrm{n}+5}, \quad n \in N$$

   We note that we can write x_n in the form

$$x_n = \frac{2+\frac{1}{\mathrm{n}}}{1+\frac{5}{\mathrm{n}}};$$

   thus x can be regarded as the quotient of $A = (2 + \frac{1}{n})$ and $B = (1 + \frac{5}{n})$. Since the later sequence consists of non-zero terms and has a limit of 1, then we can conclude from the elementary properties of limit that

$$\lim X = \frac{\lim \mathrm{A}}{\lim \mathrm{B}} = \frac{2}{1} = 2.$$

Theorem 1 *show that if* $\lim_{n\to\infty} x_n$ *exists, it must be unique.*

We need to show that if $\lim_{n\to\infty} x_n = x_1$ and $\lim_{n\to\infty} x_n = x_2$, then $x_1 = x_2$.

It follows that given any $\epsilon > 0$, we can find a positive integer $n_0(\epsilon)$ such that,

$$|x_n - x_1| < \frac{1}{2}\epsilon, \quad \text{whenever } n > n_0$$

and

$$|x_n - x_2| < \frac{1}{2}\epsilon, \quad \text{whenever } n > n_0$$

Then

$$\begin{aligned} |x_1 - x_2| = |x_1 - x_n + x_n - x_2| &\leq |x_1 - x_n| + |x_n - x_2| \\ &< \tfrac{1}{2}\epsilon + \tfrac{1}{2}\epsilon = \epsilon \end{aligned}$$

i.e., $|x_1 - x_2|$ is less than any positive ϵ (however small) and so must be zero. Thus $x_1 = x_2$.

Theorem 2 *Let* x_n *be the sequence in* R *whose nth term is given as* $x_n = \frac{1}{n}$. We shall show that

$$\lim_{n\to\infty} = 0.$$

To do this, let $\epsilon > 0$ (be a positive real number), then $\exists$ (there exists) a positive integer n_0 whose value depends on ϵ, such that

$$\frac{1}{n_0(\epsilon)} < \epsilon$$

Then, if $n > n_0$, we have

$$0 < x_n = \frac{1}{n} < \frac{1}{n_0(\epsilon)} < \epsilon$$

Thus if follows that

$$|x_n - 0| < \epsilon \text{ for } n > n_0$$

which implies that $\frac{1}{n} \to 0$ as $n \to \infty$.

Theorem 3 *Any sequence in* R *whose limit exists in* R *is bounded.*

We need to show that the following implication is true

$$[(\lim_{n\to\infty} x_n = x)].$$

Proof. Let $x = lim(x_n)$ and let $\epsilon = 1$ in the definition of limit above. Then there exists a positive integer n_0 such that

$$|x_n - x| \leq 1 \text{ for } n > n_0$$

$$\text{i.e.,} \quad |x_n| \leq |x| + 1$$

If we set $M = max\{|x_1|, |x_2|, \ldots, |x_{k-1}|, |x| + 1\}$,

$$\text{then} \quad |x_n| \leq M \;\; \text{for } n > n_0.$$

So boundedness is a necessary condition for a sequence to be convergent.

1.6 Elementary Properties of limits

Let $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$, then the following conditions hold:

1. $\lim_{n\to\infty}(x_n + y_n) = x + y$ and $\lim_{n\to\infty}(x_n - y_n) = x - y$.

2. $\lim_{n\to\infty}(x_n y_n) = xy$.

3. If in addition $y \neq 0$, and $(\forall n \in N)(y_n \neq 0)$, then

$$\lim_{n\to\infty}(\frac{x_n}{y_n}) = \frac{x}{y}.$$

We explain the results of 1 to 3 above.

1. Let $\epsilon > 0$. Choose a positive integer n_1 such that

$$|x_n - x| < \frac{\epsilon}{2} \;\; \forall n > n_1$$

Choose another positive integer n_2, such that

$$|y_n - y| < \frac{\epsilon}{2} \;\; \forall n > n_2.$$

Then for all $n > max\{n_1, n_2\}$,

$$\begin{aligned} |x_n + y_n - (x + y)| &\leq |x_n - x| + |y_n - y| \\ &< \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} = \epsilon. \end{aligned}$$

Since this can be done for arbitrary $\epsilon \leq 0$, then $(x_n + y_n) \to x + y$ as $n \to \infty$. Precisely, the same argument can be used to show that $(x_n - y_n) \to x - y$ as $n \to \infty$.

2. To show that $x_n y_n \to xy$ as $n \to \infty$, we make the estimate

$$\begin{aligned} |x_n y_n - xy| &= |x_n y_n - x_n y + x_n y - xy| \\ &\leq |x_n(y_n - y)| + |(x_n - x)y| \\ &\leq |x_n||y_n - y| + |x_n - x||y| \end{aligned}$$

Now, let there exists a positive real number M which is a $max\{|x_n|, |y_n|\}$. In addition, from the limit of x_n, y_n, we conclude that if $\epsilon > 0$ is given, then there exists positive integer n_1 and n_2 such that if $n > n_1$, then

$$|y_n - y| < \frac{\epsilon}{2M}$$

and if $n > n_2$, then

$$|x_n - x| < \frac{\epsilon}{2M}$$

Now, choose $n_0 = \max\{n_1, n_2\}$; then if $n > n_0$, we have

$$\begin{aligned} |x_n y_n - xy| &\leq M|y_n - y| + M|x_n - x| \\ &< M\{\tfrac{\epsilon}{2M} + \tfrac{\epsilon}{2M}\} = \epsilon. \end{aligned}$$

Thus, $x_n y_n \to xy$ as $n \to \infty$.

3. To show that

$$\lim_{n\to\infty} \frac{x_n}{y_n} = \frac{x}{y},$$

we estimate as follows:

$$\begin{aligned} |\tfrac{1}{y_n}x_n - \tfrac{1}{y}| &= |(\tfrac{1}{y_n}x_n - \tfrac{1}{y}x_n) + (\tfrac{1}{y}x_n) + (\tfrac{1}{y}x_n - \tfrac{1}{y}x)| \\ &\leq |\tfrac{1}{y_n} - \tfrac{1}{y}||x_n| + \tfrac{1}{|y|}|x_n - x| \\ &= \tfrac{|y-y_n|}{|y_n y|}|x_n| + \tfrac{1}{|y|}|x_n - x| \\ &= \tfrac{|y_n-y|}{|y_n y|} + \tfrac{1}{|y|}|x_n - x| \end{aligned}$$

Now let M be a positive real number such that

$$\frac{1}{M} < |y| \text{ and } |x| < M$$

It follows that there exists a natural number, n_0 such that $n > n_0$, then

$$\frac{1}{M} < |y_n| \text{ and } |x_n| < M$$

Hence if $n > n_0$, then above estimate yields,

$$|\frac{1}{y_n}x_n - \frac{1}{y}x| \leq M^3|y_n - y| + M|x_n - x|$$

Therefore, if ϵ is a pre-assigned positive real number, there exists positive integers n_1, n_2 such that $n > n_1$, then

$$|y_n - y| < \frac{\epsilon}{2M^3}$$

and if $n > n_2$, then

$$|x_n - x| < \frac{\epsilon}{2M}.$$

Letting $K = \max\{n_0, n_1, n_2\}$, we conclude that if $n > K$, then

$$|\frac{1}{y_n}x_n - \frac{1}{y}x| < M^3\frac{\epsilon}{2M^3} + M\frac{\epsilon}{2M} = \epsilon,$$

which shows that $\frac{x_n}{y_n} \to \frac{x}{y}$ as $n \to \infty$.

1.7 Convergence of Subsequence

Definition 7 *If $X = (x_n)$ is a sequence in R and if $r_1 < r_2 < r_3 \cdots < r_n < \cdots$ is a strictly increasing sequence of natural numbers, then the sequence X' in R given by*

$$(x_{r1}, x_{r2}, \ldots, x_{rn}, \ldots)$$

is called a subsequence of X.

It may be helpful to connect the notion of a subsequence with that of the composition of two functions. Let g be a function with domain N and range in N, and let g be strictly increasing in the sense that if $n < m$, then $g(n) < g(m)$. Then g defines a subsequence of $X = (x_n)$ by the formula $X \circ g = (x_{g(n)} : n \in N$.

Conversely, every subsequence of X has the form $X \circ g$ for strictly increasing function g with $D(g) = N$ and $R(g) \subseteq N$. It is clear that a given sequence has many different subsequence. Although, the next result is very elementary, it is of sufficient importance that it must be made explicit.

Lemma 1 *If a sequence X in R converges to an element x, then any subsequence of X also converges to x.*

Proof. Let V be a neighborhood of the limit element x, by definition, there exists a natural number K_v such that if $n > K_v$, then x_n belongs to V. Now let X' be a subsequence of X; say $(X' = x_{r1}, x_{r2}, \cdots, x_{rn} \cdots)$.

Since $r_n \geq n$, then $r_n \geq K_v$ and hence x_{rn} belongs to V. This proves that X' also converges to x.

Corollary 1 *If $X = (x_n)$ is a sequence which converges to an element x of R and if m is any natural number, then the sequence*

$$X' = (x_{m+1}, x_{m+2}, \cdots)$$

also converges to x.

Proof. Since X' is a subsequence of X, the result follows directly from the preceding lemma.

The preceding results have been mostly directed towards proving that a sequence converges to a given point. It is also important to know precisely what it means to say that a sequence X does not converge to x.

The next result is elementary but trivial. We leave its detailed proof to the student.

Theorem 4 *If $X = (x_n)$ is a sequence in R, then the following statements are equivalent:*

1. X does not converge to x.

2. There exists a neighborhood V of x such that if n is any natural number, then there is a natural number $m = m(n \geq n)$ such that x_m does not belong to V.

3. There exists a neighborhood V of x and a subsequence x' of X such that none of the elements of X' belong to V.

Examples

1. Let X be a sequence in R consisting of the natural numbers $X = (1, 2, 3, \ldots, n, \ldots)$. Let x be any real number and consider the neighborhood V of x consisting of the open interval $(x-1, x+1)$. Accordingly, there exists a natural number K_0 such that $x + 1 < K_0$; hence if $n \geq K_0$, it follows that $x_n = n$ does not belong to V. Therefore, the subsequence $X' = (k_0, K_0 + 1, \ldots)$ of X has no points in V, showing that X does not converge to x.

2. Let $Y = (y_n)$ be the sequence in R consisting of $Y = (-1, 1, \ldots, (-1)^n, \ldots)$. We leave it to the student to show that no point y, except possibly $y = \pm 1$, can be a limit of Y. We shall show that the point $y = -1$ is not a limit of Y; the consideration for $y = +1$ is entirely similar. Let V be the neighborhood of $y = -1$ consisting of the open interval $(-2, 0)$. Thus, if n is even, then the element $y_n = (-1^n) = +1$ does not belong to V. Therefore, the subsequence Y' of Y corresponding to $r_n = 2n, n \in N$, avoids the neighborhood V, showing that $y = -1$ is not a limit of Y.

1.8 Limit Superior and Limit Inferior

In Module 1, Unit 3, we introduced the supremum and infimum of a set of real numbers, and we have made much use of this concept since then. You will recall that we can describe the supremum of a set E of real numbers as the infimum of those real numbers which are exceeded by no element of E. In dealing with infinite sets for instance, it is often useful to relax things somewhat and allow a finite number of larger elements. Thus, if E is a bounded infinite set, it is reasonable to consider the infimum of those real numbers which are exceeded by only a finite number of elements of E.

For many purposes, however, it is important to consider a slight modification of this idea applied to sequences and not just sets of real numbers. Indeed, a sequence $X = (x_n)$ of real numbers does form a set $\{x_n\}$ of real numbers, but the sequence has somewhat more structure in that it is indexed by the set of natural numbers; hence there is a kind of ordering that is not present in arbitrary sets. As a result of this indexing the same number may occur often in the sequence while there is no such idea of "repetition" for a general set of real numbers. Once this difference is pointed out, it is easy to make the appropriate modification.

Definition 8 *Let $X = (x_n)$ be a sequence of real numbers which is bounded above, then the* limit superior*, or* greatest limit *or* upper limit *of $X = (x_n)$ is the infimum of those real numbers v with the property that there are only a finite number of natural numbers n such that $v < x_n$, and is denoted by:*

$$\lim sup X, \ \lim sup(x_n), \quad \text{or} \quad \overline{\lim}(x_n)$$

Similarly, if the real sequence X is bounded below, then the *limit inferior*, *least limit*, or *lower limit* is the supremum of those real number w, with the property that there are only a finite number of natural numbers m such that $x_m < w$, and is denoted by

$$\lim inf X, \ \lim inf(x_n), \quad \text{or} \quad \underline{\lim}(x_n)$$

Theorem 5 *Let $X = (x_n)$ be a sequence in R which is bounded above, then the following statements are equivalent:*

1. $x^* = \lim sup(x_n)$.
2. If $\epsilon > 0$, then there are only a finite number of positive integers n such that $x^* + < x_n$, but there are an infinite number such that $x^* - \epsilon < x_n$.

3. If $v_m = sup\{x_n : n \geq m\}$, then $x^* = inf\{v_m : m \geq 1\}$.

4. If $v_m = sup\{x_n : n \geq m\}$, then $x^* = \lim(v_m)$

We now establish the basic algebraic properties of the superior and inferior limits of a sequence. For simplicity, we shall assume that the sequences are bounded, although some extensions are clearly possible.

1.8.1 Algebraic Properties

let $X = (x_n)$ and $Y = (y_n)$ be bounded sequences of real numbers. Then the following relations hold:

1. $\lim inf(x_n) \geq \lim sup(x_n)$.

2. If $c > 0$, then $\lim inf(cx_n) = c \lim inf(x_n)$ and $\lim sup(cx_n) = c \lim sup(x_n)$.

3. If $c \leq 0$, then $\lim inf(cx_n) = c \lim sup(x_n)$ and $\lim sup(cx_n = c \lim inf(x_n))$.

4. $\lim inf(x_n) + \lim inf(y_n) \geq \lim ing(x_n + y_n)$.

5. $\lim sup(x_n + y_n) \leq \lim sup(x_n) + \lim sup(y_n)$.

6. If $x_n \leq y_n$ for all n, then $\lim inf(x_n) \leq \lim inf(y_n)$ and also $\lim sup(x_n) \leq \lim sup(y_n)$.

Proof.

1. If $w < \lim inf(x_n)$ and $v > \lim sup(x_n)$, the there are infinitely many natural numbers n such that $w \leq x_n$, while there are only a finite number such that $v < x_n$. Therefore, we must have $w \leq v_1$ which proves 1.

2. If $c > 0$, then multiplication by c preserves all inequalities of the form $w \leq x_n$ etc.

3. If $c \leq 0$, then multiplication by c reverses inequalities and converts the limit superior into the limit inferior, and conversely.

Statements 4 is dual to 5 and can be derived directly from 5 or proved by using the same type of argument. To prove 5, let $v > \lim sup(x_n)$ and $u > \lim sup(y_n)$; by definition, there are only a finite natural number of natural numbers n such that $y_n > u$. Therefore, there can be only a finite number of n such that

$$x_n + y_n > v + u$$

Showing that $\lim sup(x_n + y_n) \leq v + u$.

This proves statement 5.

We now prove the second assertion in 6. If $u > \lim sup(y_n)$, then there can be only a finite number of natural numbers n such that $u < y_n$. Since $x_n \leq y_n$, then $\lim sup(x_n) \leq u$, and so $\lim sup(x_n) \leq \lim sup(y_n)$.

1.9 Conclusion

Limits holds a central position which permeates the whole of mathematical analysis. It forms the foundation for the theory of approximation, convergence of sequence and continuity, and differential and integral calculus. Though, the concept of limit may not be simple to comprehend at first glance, however, we have tried as much as possible to present our discussion in a progressive manner to make your understanding easier while elaborate examples and illustrations are given for interesting reading. Our next unit shall consists discussion on classes of convergence sequences such as monotone, and Cauchy sequences.

1.10 Summary

We commenced this unit with elaborate introduction followed by a gradual foray into sequences and its properties to prepare the ground for the concept of limit, its properties and convergence of sequences. Various theorems that provides the richness of limits and convergence of sequence were examined together with elaborate examples. Also important of mention is the theory of subsequence, as well as limit superior and limit inferior which were treated towards the end of the unit.

1.11 Tutor Marked Assignments

1. If $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$. Prove that $\lim_{n\to\infty}(a_n + b_n) = a + b$.

2. Prove that a convergent sequence is bounded and hence show that if a sequence is unbounded then is diverges.

3. Use the definition of the limit of a sequence to prove that $\lim_{n\to\infty} \frac{3n+2}{4-5n} = \frac{-3}{5}$.

1.12 References

[1.] Anderson, K.w., and D.W. Hall, *Sets, Sequences, and Mappings*, John Wiley, New York, 1963.

[2.] Hull, D.W. and G.L. Spencer II, *Elementary Topology*, John Wiley, New York, 1955.

[3.]Krantz, S., *Real Analysis and Foundations*, CRC Press, 1991.

[4.] Roydem, H.L., *Real Analysis*, Macmilan, New York, 1963.

[5.] Spiegel, M.R., *Theory and Problems of Real Variables*, McGraw-Hill, New York, 1969.

Unit 2: Classes of Convergence Sequences in R

1.1 Introduction

In the previous unit, the main method available for showing that a sequence is convergence is to establish the existence of the limit. However, the use of this tool has a limitation that we must already know (or at least suspect) the correct value of the limit and we merely verify that our suspicion is correct.

There are many instances, however, where there is no obvious candidate for the limit of a given sequence, even though a preliminary analysis has led to the belief that a convergence does take place. In this unit, we shall give some results which are deeper than those in the preceding unit, and which can be used to establish the convergence of a sequence when no particular element presents itself as a candidate for the limit. In fact, we do not even determine the exact value of the limit of the sequence.

1.2 Objective

Our goal here is to explore other ways in which the convergence of a sequence can be determined without establishing its limits; the limit might even be difficult to establish.

1.3 Monotone Sequence

In this section, we shall consider a particular class of sequence which often occur in applications.

Definition 9 *We shall adopt an incremental approach in our definition:*

1. *A sequence of real numbers* $X = (x_n)$ *is said to be* monotone increasing *if*

$$x_1 \leq x_2 \leq \cdots \leq x_n \leq x_{n+1} \leq \cdots$$

which can be denoted by

$$x_n \leq x_{n+1}, \quad \forall x \in N.$$

2. *A sequence of real numbers* $X = (x_n)$ *is said to be* monotone decreasing *if*

$$x_1 \geq x_2 \geq \cdots \geq x_n \geq x_{n+1} \geq \cdots$$

which can be denoted by

$$x_n \geq x_{n+1}, \quad \forall n \in N$$

3. *A sequence of real numbers* $X = (x_n)$ *is said to be* monotone *if it is either increasing or decreasing.*

Examples 5.1

1. $x_n = n^2$ is monotone increasing.

2. $x_n = \frac{1}{n}$ is monotone decreasing.

3. $x_n = (-1)^n \frac{1}{n}$ is not monotone.

The next theorem is one of the main theorems in the theory of converging sequences.

Theorem 6 *Let* $X = (x_n)$ *be a sequence of real numbers which is monotone increasing, then the sequence* X *converges if and only if it is bounded, in which case*

$$\lim(x_n) = sup\{x_n\}.$$

Proof. It was established in unit 1 that a convergence sequence is bounded. If $x = \lim(x_n)$ and $\epsilon \geq 0$, there exists a positive integer $n_0(\epsilon)$ such that if $n > n_0$, then

$$x - \epsilon \leq x_n \leq x + \epsilon$$

Since X is monotone, this relation yields

$$x - \epsilon \leq sup\{x_n\} \leq x + \epsilon,$$

it follows that

$$|x - sup\{x_n\}| \leq \epsilon.$$

Since this holds for all $\epsilon > 0$, then the

$$\lim(x_n) = x = sup\{x_n\}.$$

Conversely, suppose that $X = (x_n)$ is a bounded monotone increasing sequence in R; according to the supremum principle in module 1, unit 3, the supremum $x^* = sup\{x_n\}$ exists; we shall show that it is the limit of X. Since x^* is an upper bound of the elements in X, then

$$x_n \leq x^*, \quad \text{for } n \in N.$$

Since x^* is the supremum of X, if $\epsilon > 0$ the number $x ** - \epsilon$ is not an upper bound of X, thus, there exists a positive integer such that

$$x^* - \epsilon < x_{n_0}.$$

In view of the monotone character of X, if $n > n_0$, then

$$x^* - \epsilon < x_n \leq x^*$$

which follows that

$$|x_n - x^*| < \epsilon.$$

recapitulating, the number

$$x^* = sup\{x_n\}$$

has the property that given $\epsilon > 0$, there exists a positive integer $n_0(\epsilon)$ such that

$$|x_n - x^*| < \epsilon \quad \text{whenever } n > n_0.$$

This shows that

$$x^* = \lim X.$$

Corollary 2 *Let $X = (x_n)$ be a sequence in R, which is monotone decreasing, then the sequence X converges if and only if it is bounded, in which case*

$$\lim(x_n) = inf\{x_n\}.$$

Proof. Let $y_n = -x_n \quad$ for $n \in N$.
Then the sequence $Y = (y_n)$ is readily seen to be monotone increasing sequence. Moreover, Y is bounded if and only if X is bounded. Therefore, the conclusion follows from Theorem 6 above.

Examples 5.2

Let $X = (x_n)$ be the sequence of real numbers defined by

$$x_n = \frac{1}{n}.$$

Recall that if x_n is monotone increasing, then

$$x_1 \leq x_2 \leq \cdots \leq x_n \leq x_{n+1} \leq \cdots$$

and if (x_n) is monotone decreasing, then

$$x_1 \geq x_2 \geq \cdots \geq x_n \geq x_{n+1} \geq \cdots$$

Hence from observation of (x_n), we find that

$$1 < \frac{1}{2} < \frac{1}{3} < \cdots < \frac{1}{n} \ \ \forall n \in N$$

which is a decreasing monotone sequence. Also, we see that $x_n > 0 \ \forall n \in N$. It therefore follows from Corollary 2 that the sequence $X = (\frac{1}{n})$ converges. (Of course we know that the limit of X is); but the existence of the limit follows even if we are not able to evaluate accurately the $inf\{x_n\}$.)

1.4 Cauchy Sequence

The monotone convergence sequence as discussed above is very useful and important, but it has a drawback in that it applies only to sequences which is monotone. It therefore follows that we need to find a condition which will imply convergence in R without using the monotone property. This desired condition is the *Cauchy Criterion*, which we shall introduce subsequently.

Definition 10 *A sequence $X = (x_n)$ in R is said to be a Cauchy sequence if given any $\epsilon > 0$, $\exists$ a positive integer $n_0(\epsilon)$ such that*

$$|x_m - x_n| < \epsilon \quad \textit{whenever} \ \ m > n_0,\, n > n_0.$$

Next, we shall show that every convergent sequence in R is a Cauchy sequence.

Lemma 2 *If $X = (x_n)$ is a convergent sequence in R, then X is a Cauchy sequence.*

Proof. If $x = \lim X$; then given $\epsilon > 0$, $\exists$ a positive integer $n_0(\frac{\epsilon}{2})$ such that

$$|x_n - x| < \frac{\epsilon}{2} \ \ \forall\, n > n_0.$$

Now, let $m, n > n_0$, then we have

$$|x_m - x_n| \leq |x_m - x| + |x - x_n| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Hence the convergent sequence X is a Cauchy sequence.

Lemma 3 *A Cauchy sequence in R is bounded.*

Proof. Let $X = (x_n)$ be a Cauchy sequence, and let $\epsilon = 1$, then there exists a positive integer $n_0(1)$, such that for $n > n_0$, then

$$|x_m - x_n| < 1.$$

Thus,

$$|x_n| < |x_m + 1| \ \ \forall\, n \in n_0.$$

Therefore, if we set

$$M = sup\{|x_1|, |x_2|, \ldots, |x_{m-1}|, |x_m| + 1\}.$$

then we have

$$x_n \leq M \ \ \forall\, n \in N.$$

Thus the Cauchy sequence X is bounded.

Example 5.3

Let $X = (x_n)$ be the sequence in R defined by

$$x_1 = 1,\, x_2 = 2,\, \ldots,\, x_n = \frac{\mathrm{x_{n-2} + x_{n-1}}}{2} \quad \text{for } n > 2.$$

Now, for $n = 3, 4, 5, 6, \ldots$, it can be shown by induction that

$$1 \leq x_n \leq 2 \quad \text{for } n \in N,$$

but the sequence X is neither monotone decreasing nor increasing. (Actually, the terms with odd subscript form an increasing sequence, while those with even subscript form a decreasing sequence). Since the terms in the sequence are formed by averaging, it is readily seen that

$$|x_n - x_{n+1}| = \frac{1}{2^{\mathrm{n}-1}} \ \ \forall n \in N.$$

Thus, if $m > 0$, we obtain

$$\begin{aligned} |x_n - x_m| \leq |x_m - x_{n+1}| + \cdots + |x_{m-1} + x_m| &= \tfrac{1}{2^{n-1}} + \cdots + \tfrac{1}{2^{m-2}} \\ &= \tfrac{1}{2^{n-1}}(1 + \tfrac{1}{2} + \cdots + \tfrac{1}{2^{m-n-1}}) \\ &< \tfrac{1}{2^{n-2}}. \end{aligned}$$

Given $\epsilon > 0$, if n is chosen so large that

$$\frac{1}{2^{\mathrm{n}}} < \frac{\epsilon}{4}$$

and if $m \geq n$, it follows that

$$|x_n - x_m| < \epsilon.$$

Therefore, X is a Cauchy sequence in R, and by Cauchy Criterion, the sequence X converges to a number x. To evaluate the limit, we note that on taking the limit, the rule of definition yields the validity but uninformative result $x = \frac{1}{2}(x + x) = x$.

1.5 Conclusion

It is observed that despite importance of the monotone convergence theorem, it still has a drawback – it is applicable to only sequences which are monotone. Thus, in order to find a condition which will imply convergence in R without using the monotone property, the Cauchy Criterion is introduced. So, once a sequence is not a monotone, then the Cauchy Criterion will be handy to establish the convergency or otherwise of the sequence. This concludes the second part of Module 2. Our next Module will be concentrating on the limits and continuity of functions.

1.6 Summary

This unit covered the two main classes of sequences in R, which are monotone sequences and Cauchy sequences. Examples were given to illustrate their properties and usefulness.

1.7 Tutor Marked Assignments

1. Let $Y = (y_n)$ be the sequence in R defined by

$$y_1 = 1, \; y_{n+1} = \frac{2y_n + 3}{4} \quad \text{for } n \in N.$$

   Show that Y is monotone and hence converges.

2. If $X = (x_n)$ is the sequence defined in R by

$$x_n = \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n} \quad \text{for } n \in N.$$

   Show that the sequence X is a Cauchy sequence in R.

1.8 References

[1.] Anderson, K,W., and D.W. Hull, *Sets, Sequences, and Mappings*, John Wiley, New York, 1963.

[2.] Graves, L.M., *Theory of Foundations of Real Variables*, Second Edition, McGraw-Hill, New York, 1968.

[3.] Krantz, S., *Real Analysis and Foundations*, CRC Press, 1991.

[4.] Roydem, H.L., *Real Analysis*, MacMillan, New York, 1963.

[5.] Spiegel, M.R., *Theory and Problems of Real Variables*, McGraw-Hill, New York, 1969.

PART 3: LIMITS & CONTINUITY OF SEQUENCES OF FUNCTIONS

Unit 1: Limits of Functions

1.1 Introduction

Elementary analysis in mathematics deals with several different types of limit operations. We have already discussed the limits and convergence of sequences of real numbers in our previous modules, we shall bring in the limiting operations of functions and its properties in this unit.

We shall let f be a function with domain D contained in R and values in R, and shall consider the limiting character of f at any point in D.

1.2 Objective

Although it is not easy to draw a definite borderline, it is fair to characterize analysis as that part of mathematics where systematic use is made of various limiting concepts. The notion for the study of the limits of functions is to provide adequate basics connected with limiting operations in the theory of the derivative of functions and real-valued functions in general integral theory.

Definition 11 *A number l is said to be the limit of $f(x)$ as x approaches a if for every $\epsilon > 0$, $\exists$ a $\delta > 0$ such that*

$$|f(x) - l| < \epsilon \quad \text{whenever } 0 < |x - a| < \delta.$$

In such a case, we write

$$\lim_{x \to a} f(x) = l.$$

For this definition to be valid, we need to show that

$$0 < |x - a| < \delta \ \Rightarrow \ |x - a| < \epsilon.$$

Hence one can choose $\delta = \epsilon$, such that $\forall x \in R$, then

$$0 < |x - a| < \delta \ \Rightarrow \ |x - a| < \epsilon.$$

Example 6.1

Let $f : R \mapsto R$ be defined by $f(x) = x^2$, thus for any $a \in R$

$$\lim_{x \to a} x^2 = a^2$$

Proof. Let $a \in R$ be arbitrary and let $\epsilon > 0$. We need to show that

$$0 < |x - a| < \delta \ \text{ implies that } \ |x^2 - a^2| < \epsilon.$$

We can restrict ourselves to $|x - a| < 1$. Then

$$|x - a| \leq |x| + |a| < 1 + 2|a|,$$

and

$$|x - a| \leq |x| + |a| < 1 + 2|a|.$$

Note that

$$|x^2 - a^2| = |x - a||x + a| < (1 + 2|a|)|x - a|.$$

Now, choose $\delta = \ \min \{1, \frac{\epsilon}{1+2|a|}\}$. Then $\forall x \in R$

$$0 < |x - a| < \delta \ \Rightarrow \ |x^2 - a^2| < \epsilon.$$

Example 6.2

Using $f(x) = x^2$ in Example 6.1 above, show that $\lim_{x \to 2} f(x) = 4$.

We must show that given any $\epsilon > 0$, there exists $\delta > 0$ (depending on ϵ in general) such that

$$|x^2 - 4| < \epsilon \ \text{ whenever } \ 0 < x - 2 < \delta.$$

Choose $\delta \leq 1$, so that

$$0 < |x - 2| < \delta \leq 1.$$

Note that

$$\begin{aligned} |x^2-4| = |(x-2)(2+2)| = |x-2||x+2| &< \delta|x+2| \\ &= \delta|(x-2)+4| \\ &\leq \delta(|x-2|+4) \\ &< 5\delta \end{aligned}$$

Now, choose $\delta =$ min $\{1, \frac{\epsilon}{5}\}$. Then we have,

$$|x^2-4| < \epsilon \text{ whenever } 0 < |x-2| < \delta.$$

Example 6.3

Prove that $\lim_{x\to 0} x\sin(\frac{1}{2}) = 0$.

Proof. We must show that given any $\epsilon > 0$, we can find $\delta > 0$ such that

$$|x\sin(\frac{1}{x}) - 0| < \epsilon \text{ whenever } 0 < |x-0| < \delta.$$

If $0 < |x| < \delta$, then

$$|x\sin(\frac{1}{x})| = |x|\sin(\frac{1}{x}) \leq |x| < \delta.$$

Since $\sin(\frac{1}{x}) \leq 1$ for $x \neq 0$. Making the choice $\delta = \epsilon$, we see that

$$|x\sin(\frac{1}{x})| < \epsilon \text{ whenever } 0 < |x| < \delta,$$

completing the proof.

1.3 Elementary Properties

Limits of functions exhibit many properties similar to those of limits of sequences. First, let us prove the *uniqueness*

Theorem 7 *Let $f : R \mapsto R$. If $\lim_{x\to a} f(x) = A$ and $\lim_{x\to a} f(x) = B$, then $A = B$.*

Proof. Let $x_n \to a$ as $n \to \infty$. Then $f(x_n) \to A$ as $n \to \infty$ and $f(x_n) \to B$ as $n \to \infty$. From the uniqueness of the limit of a sequence (see module 2, unit 1). From the uniqueness of the limit of a sequence, it follows that $A = B$.

The following properties of the limit of a function is similar to those of the limit of a sequence. The proofs are left as exercises.

Theorem 8 *Let* $\lim_{x\to a} f(x) = A$ *and* $\lim_{x\to a} g(x) = B$*. Then the following relations hold:*

1. $\lim_{x\to a}(f(x) + g(x)) = A + B$
2. $\lim_{x\to a}(f(x)g(x)) = AB$
3. $\lim_{x\to a}(f(x)/g(x)) = A/B, \quad \text{provided } B \neq 0$

We leave the proofs of the above theorem as exercise. However, using the theorem, one can easily compute limits of the following function.

Example 6.3

$$\begin{aligned} \lim_{x\to 2} \frac{x^3 + 2x^2 - 7}{2x^3 - 4} &= \frac{\lim\limits_{x\to 2} x^3 + \lim\limits_{x\to 2} 2x^2 - \lim\limits_{x\to 2} 7}{\lim\limits_{x\to 2} 2x^3 - \lim\limits_{x\to 4}} \\ &= \frac{8 + 8 - 7}{16 - 4} = \frac{3}{4} \end{aligned}$$

1.4 One-Sided Limits

Definition 12 *The definition of one-sided limits consists of two parts:*

1. Let f be defined on our interval $(c, d) \subset R$. We say that $f(x) \to b$ as $x \to a^+$ and write

$$\lim_{x\to a^+} f(x) = b$$

   if given $\epsilon > 0$, $\exists\, \delta > 0$ such that $\forall\, x \in (c, d)$ then

$$0 < x - a < \delta \ \Rightarrow\ |f(x) - b| < \epsilon$$

2. Let f be defined on an interval $(c, d) \subset R$. We say that $f(x) \to b$ as $x \to a^-$ and write

$$\lim_{x\to a^-} f(x) = b$$

   if given $\epsilon > 0$, $\exists\, \delta > 0$ such that $\forall\, x \in (c, d)$ then

$$-\mathrm{d} < x - a < 0 \ \Rightarrow\ |f(x) - b| < \epsilon$$

1.5 Conclusion

It is observed that the theory of limits and sequences in R discussed in the previous module is analogous to the limits of functions except in the area of continuity which we shall consider in the next unit. It is therefore advised that students should be comfortable with the theory of limits and convergence before attempting this unit in order to prepare you for easy comprehension. Our next unit shall be concerned with continuity, boundedness and uniform continuity of limits of functions.

1.6 Summary

Our coverage in this unit include limits of functions and its properties. In as much as continuity of limits of functions is very closely tied to this unit, we have decided to leave it out in order to devote adequate time for its treatment.

1.7 Tutor Marked Assignments

1. Let
$$f(x) = \begin{cases} x^2, & x \neq 2 \\ 0, & x = 2. \end{cases}, \quad \text{prove that } \lim_{x \to 2} f(x) = 4$$

2. Let
$$f(x) = \begin{cases} \frac{|x-3|}{x-3}, & x \neq 3 \\ 0, & x = 3 \end{cases}, \quad \text{hence find } \lim_{x \to 3} f(x).$$

1.8 References

[1.] Anderson, K,W., and D.W. Hull, *Sets, Sequences, and Mappings*, John Wiley, New York, 1963.

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Unit 2: Continuous Functions

1.1 Introduction

In the two preceding units, we discussed the limits/convergence of sequence and functions in R, in this present unit, we shall consider continuous functions. We shall equally discuss the notion of boundedness and uniform continuity of functions. Similar to our treatment on limits of functions in the previous unit, we shall consider functions which have their common domain in the Cartesian space R^p, and their range in R^q. We shall use the same symbol to denote the algebraic operations. As usual, we shall include as many examples as possible in order to deepen your understanding.

1.2 Objective

Continuity of functions, boundedness and uniform continuity are important class of functions in analysis. If you must make any appreciable progress in the study of mathematical analysis such that you would find in topology and integral theory, this topic is a must and it takes it root from our previous units. At the end of the unit, we expect students to be able to understand the conditions necessary for a function to be continuous and uniform continuity of functions.

Definition 13 *Let $a \in R$. A function $f : R \mapsto R$ defined in a neighborhood of a is said to be continuous at a point a, if given $\epsilon > 0$, there exists a $\delta > 0$ such that*

$$|f(x) - f(a)| < \epsilon \quad \textit{whenever} \quad |x - a| < \delta.$$

Equivalently, we can say that f (or $f(x)$) is continuous at a if the following relations hold:

1. $\lim_{x \to a} f(x)$ exists,

2. $f(a)$ exists and

3. $\lim_{x\to a} f(x) = f(a)$.

Definition 14 *A function f is continuous at a point $a \in R$ if f is defined on an interval (a, d) containing a and given any $\epsilon > 0$ there exists a $\delta > 0$ such that $\forall\, x \in (c, d)$*

$$|x - a| < \delta \;\; \Rightarrow \;\; |f(x) - f(a)| < \epsilon.$$

Note the difference between this definition and the definition of the limit − the function f has to be defined at a point a. Using the above definition, it is easy to formulate what it means that a function is discontinuous at a point a.

Definition 15 *A function is discontinuous at a point $a \in R$ if f is not defined on any neighborhood (c, d) containing a, or if given any $\epsilon > 0$, there exists $x \in D(f)$ such that $\forall\, \delta > 0$, then*

$$(|x - a| < \delta) \wedge (|f(x) - f(a)| \geq \epsilon).$$

Examples 7.1

Investigate the continuity of

1. $f(x) = x^2$ at $x = 2$.

2. $f(x) = x \sin(\frac{1}{x})$ at $x = 0$.

Solutions

1. By example 6.2 in unit 1, we have

$$\lim_{x\to 2} f(x) = 4 = f(2),$$

   i.e., the limit of $f(x)$ as $x \to 2$ equally the value of $f(x)$ at $x = 2$, and so f or $f(x)$ continuous at $x = 2$.

2. Since $f(x)$ is not defined for $x = 0$, $f(x)$ cannot be continuous at $x = 0$.

Remark 1 *1. Let $c \in R$. Let $f : R \mapsto R$ be defined by $f(x) = c$ for any $x \in R$. Then f is continuous at any point in R.*

2. *Let $f : R \mapsto R$ be defined by $f(x) = x$ for any $x \in R$. Then f is continuous at any point in R.*

The following theorem easily follows from the definition of continuity and properties of limits.

Theorem 9 *Let f and g be continuous at $a \in R$. Then the following relations hold:*

1. *$f + g$ is continuous at a.*

2. *fg is continuous at a.*

3. *f/g is continuous at a, provided $g(a) \neq 0$.*

Based on the theorem above, and the Remark 1, we conclude that

$$f(x) = \frac{a_0x^n + a_1x^{n-1} + \cdots + a_n}{b_0x^m + b_1x^{m-1} + \cdots + b_m}$$

is continuous at every point of its domain of definitions.

Theorem 10 *Let g be continuous at $a \in R$, f be continuous at $b = g(a) \in R$. Then $f \circ g$ is continuous at a.*

Proof. Let $\epsilon > 0$. Since f is continuous at b, then $\exists\, \delta > 0$ such that $\forall\, y \in D(f)$, then

$$|y - b| < \delta \ \Rightarrow\ |f(y) - f(b)| < \epsilon.$$

If we take $\delta > 0$, from the continuity of g at a, $\exists\, \gamma > 0$ such that $\forall\, x \in D(g)$, then

$$|x - a| < \gamma \ \Rightarrow\ |g(x) - g(a)| < \delta.$$

From the above, it follows that given $\epsilon > 0$, $\exists\, \gamma > 0$ such that $\forall\, x \in D(g)$, then

$$|x - a| < \gamma \ \Rightarrow\ |f(g(x)) - f(g(a))| < \epsilon.$$

This proves the continuity of $f \circ g$ at a.

Another useful characterization of continuity of a function f at a point a is the following: f is continuous at a point $a \in R$ if and only if

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = f(a).$$

which means that the one-sided limit exist, are equal, and equal the value of the function at a.

Theorem 11 *Let f be continuous at $a \in R$, such that $f(a) < B$. Then, there exists a neighborhood of a such that $f(x) < B$ for all the points x from the neighborhood.*

Proof. Take $\epsilon = B - f(a)$, in Remark 1 above, then let there exists $\delta > 0$ such that for all $x \in (c, d)$, then

$$|x - a| < \delta \;\Rightarrow\; |f(x) - f(a)| < \epsilon.$$

Note: A similar fact is true for the case of $f(x) > B$.

1.3 Bounded Functions

Definition 16 *Let $f : D(f) \mapsto R$, $D(f) \subset R$, f is called* bounded *if $Ran(f)$ is a bounded subset of R.*

Example 7.2

Let $f : R \mapsto R$ be defined by

$$f(x) = \frac{1}{1 - x^2}$$

Then $Ran(f) = (0, 1]$, so f is bounded.

Definition 17 *Let $A \subset D(f)$. f is called bounded above on A if there exists $M \in R$ such that*

$$f(x) \leq M \;\; \forall x \in A.$$

Note: Boundedness below and boundedness above on a set is defined analogously.

Theorem 12 *If f is continuous at a point a, then there exists $\delta > 0$, such that f is bounded on the interval $(a - \delta, a + \delta)$.*

Proof. Since $\lim_{x \to a} f(x) = f(a)$, if we take $\epsilon = 1$, then there exists $\delta > 0$ such that $\forall x \in D(f)$,

$$|x - a| < \delta \;\Rightarrow\; |f(x) - f(a)| < 1.$$

So, on the interval $(a - \delta, a + \delta)$

$$f(a) - 1 < f(x) < f(a) + 1.$$

1.4 Continuous Functions on a Closed Interval

In the previous section, we dealt with functions which are continuous at a point. Here we shall consider functions which are continuous at every point of an interval [a,b]. In this case, we are concerned with functions which are continuous at [a,b]. In comparison with the previous section, we shall be interested here on the behavior of such functions.

Definition 18 *Let $[a,b] \subseteq R$ be a closed interval, if a function with $f : D(f) \mapsto R$, and with $[a,b] \subseteq D(f)$. Then we say that f is a continuous function on [a,b] if*

1. *f is continuous at every point of (a,b), and*

2. $\lim_{x\to a^+} f(x) = f(a)$; $\lim_{x\to b^-} f(x) = f(b)$.

The following theorem is said to exhibit the intermediate-value property of continuous function.

Theorem 13 The Intermediate-Value Theorem *Let f be a continuous functions on a closed interval $[a,b] \subset R$. Suppose that $f(a) < f(b)$, then $\forall\, \beta \in [f(a), f(b)]$ there exists $x \in [a,b]$ such that $f(x) = B$.*

Proof. If $\beta = f(a)$ or $\beta = f(b)$, then there is nothing to prove. However, if we fix $\beta \in (f(a), f(b))$, let us introduce the set

$$A = \{x \in [a,b] : f(x) < \beta\}$$

Then set A is not empty since $a \in A$. The set A is bounded above by b. Therefore, there exists $\alpha = supA$.

Our aim now is to prove that $f(\alpha = B)$. We shall do that by ruling out two possibilities: $f(\alpha) < \beta$ and $f(\alpha) > \beta$.

First, let us assume that $f(\alpha) < \beta$. Then by Theorem 11, we have that, there exists $\delta > 0$ such that $\forall\, x \in D(f)$, then

$$x \in (\alpha - \delta, \alpha + \delta) \;\; \Rightarrow \;\; f(x) < \beta.$$

Therefore, let there exists $x_1 \in R$ such that

$$(x \in (\alpha, \alpha + \delta)) \wedge (f(x) < \beta).$$

In other words, there exists $x_1 \in R$ such that

$$(x_1 > \alpha) \wedge (x_1 \in A)$$

This contradicts the fact that α is an upper bound of A.

Next, let us assume that $f(\alpha) > \beta$. Then by the Note after Theorem 11, we have that $\exists\,\delta > 0$ such that

$$D(f) \cap (\alpha - \delta, \alpha + \delta) \subset A^c \text{ } (A^c \text{ is the complement to A}).$$

This contradicts the fact that $\alpha = supA$.

The next theorem establishes boundedness of functions continuous on an interval.

Theorem 14 *Let f be continuous on [a,b]. Then f is bounded on [a,b].*

Proof. Let us introduce a set

$$A = \{x \in [a, b] : f \text{ is bounded on } [a, b]\}$$

Note that

1. $A \neq \emptyset$, since $a \in A$,
2. A is bounded (by b).

Therefore, there exists $\alpha = supA$.

Our first task shall be to prove that $\alpha = b$.

First, we prove that $\alpha > a$. Indeed, if we let $\alpha = a$, then by continuity of f at a, it follows that $\exists\,\delta > 0$ such that $\forall\, x \in D(f)$

$$0 \leq x - a < \delta \;\; \Rightarrow \;\; |f(x) - f(a)| < 1$$

Hence, the function f is bounded on $[a, a+\delta]$. Therefore, $\exists\, x_1 \in (a, a+\delta)$ such that f is bounded on $[a, x_1]$, which proves that $\alpha > a$.

Now suppose that $\alpha < b$. Then by Theorem 12, there exists $\delta > 0$ such that f is bounded on $[a, x_1]$.

Also, it follows that $\exists x_2 \in (\alpha, \alpha + \delta)$ such that f is bounded on $[a, x_1]$.

Therefore, f is bounded on $[a, b]$, where $x_2 > \alpha$ which contradicts the fact that α is the supremum of A. This proves that $\alpha = b$.

Note that this does not complete the proof since supremum may not belong to a set, i.e., it is possible that $\alpha \notin A$.

From the continuity of f at b, it follows that $\exists\, \delta_1 > 0$ such that f is bounded on $(b - \delta_1, b]$.

By the definition of supremum, $\exists\, x_1 \in (b - \delta_1, b])$ such that f is bounded on $[a, x_1]$.

Therefore, f is bounded on $[a, b]$.

The last theorem asserts that the range of a continuous function is restricted to a closed interval is a bounded subset in R, so that there exists supremum and infimum of it.

The next theorem asserts that these values of the function at this points equal to the supremum (infimum).

Theorem 15 *Let f be continuous on $[a, b] \subset R$. Then $\exists\, y \in [a, b]$ such that $\forall\, x \in [a, b]$,*

$$f(x) \leq f(y),$$

$$i.e., \quad f(y) = \max_{x \in [a,b]} f(x)$$

Similarly, let $\exists\, z \in [a, b]$, such that

$$f(x) \geq f(z).$$

Proof. Let us introduce the set of values of f on $[a, b]$, then

$$F = \{f(x) : x \in [a, b]\}$$

We know that $F \neq \emptyset$ and by Theorem 14, F is bounded. Therefore, its supremum exists.

Now denote $\varphi = supF$. Our aim is to prove that there exists $y \in [a, b]$ such that $f(y) = \varphi$. We prove this by contradiction.

Suppose that, $\forall\, x \in [a, b]$, $f(x < \varphi)$.

Define the following functions

$$g(x) = \frac{1}{4f(x)}, \;\; x \in [a, b].$$

Since the denominator is never zero on $[a, b]$, we conclude that g is continuous and by Theorem 14, is bounded on $[a, b]$. At the same time, by the definition of supremum, let $\forall\, \epsilon > 0$, $\exists\, x \in [a, b]$ such that

$$f(x) > \varphi - \epsilon$$

in other words, $\varphi - f(x) < \epsilon$, and so $g(x) > \frac{1}{\epsilon}$.

This proves that, $\forall\, \epsilon > 0$, $\exists\, x \in [a, b]$ such that

$$g(x) > \frac{1}{\epsilon}.$$

Therefore g is unbounded on $[a, b]$, which is a contradiction.

1.5 Uniform Continuity

Definition 19 *Let f be a function defined on a set $A \subset R$. $f : R \mapsto R$ is said to be* uniformly continuous *on A if given any $\epsilon > 0$, there exists a number $\delta > 0$ such that*

$$|f(x_1) - f(x_2)| < \epsilon \quad \textit{whenever } |x_1 - x_2| < \delta \ \ \forall x_1, x_2 \in A.$$

Theorem 16 *Let f be defined and continuous on a closed interval $[a, b]$. Then f is uniformly continuous on $[a, b]$. We shall leave this theorem without proof as an exercise.*

The next example shows that the property of uniform continuity is stronger than the property of continuity.

Example 7.3

The function $f(x) = \frac{1}{x}$ is continuous on (0,1) but not uniformly continuous. Indeed, take

$$x_n = \frac{1}{n}; \quad y_n = \frac{1}{n+1}.$$

Example 7.4

The function $f(x) = \sqrt{x}$ is uniformly continuous on $[1, \infty)$. Indeed,

$$\begin{aligned} |f(x_1) - f(x_2)| < |\sqrt{x_1} - \sqrt{x_2}| &= \frac{x_1 - x_2}{\sqrt{x_1} + \sqrt{x_2}} \\ &\leq |x_1 - x_2|. \end{aligned}$$

Example 7.5

Show that $f(x) = x^2$ is uniformly continuous on $0 < x < 1$.

We must show that given any $\epsilon > 0$, we can find $\delta > 0$ such that

$$|x^2 - x_0^2| < \epsilon \quad \text{whenever} \quad |x - x_0| < \delta,$$

where δ depends only on ϵ and not on x_0, where $0 < x_0 < 1$.

If x and x_0 are any points on $0 < x < 1$, then

$$|x^2 - x_0^2| = |x + x_0||x - x_0| < |1 + 1||x - x_0| = 2|x - x_0|.$$

Thus, if $|x - x_0| < \delta$, it follows that

$$|x^2 - x_0^2| < 2\delta.$$

Choose $\delta = \frac{\epsilon}{2}$, then we see that

$$|x^2 - x_0^2| < \epsilon \quad \text{whenever} \quad |x - x_0| < \delta,$$

such that δ depends only on ϵ and not on x_0. Hence, $f(x) = x^2$ is uniformly continuous on $0 < x < 1$.

Example 7.6

Show that $f(x) = \frac{1}{x}$ is not uniformly continuous on $0 < x < 1$.

Suppose that $f(x)$ is uniformly continuous o the given interval . Then for any $\epsilon > 0$, we should be able to find δ, say between 0 and 1, such that

$$|f(x) - f(x_0)| < \epsilon \quad \text{whenever} \quad |x - x_0| < \delta$$

or we can say that

$$|x - x_0| < \delta \quad \Rightarrow \quad |f(x) - f(x_0)| < \epsilon$$

for all x and x_0 in the interval.

Let $x = \delta$ and $x_0 = \frac{\delta}{1+\epsilon}$. Then

$$|x - x_0| = |\delta - \frac{\delta}{1+\epsilon}| = \frac{\epsilon}{1+\delta}\delta < \delta$$

However,

$$|\frac{1}{x} - \frac{1}{x_0}| = |\frac{1}{\delta} - \frac{1+\delta}{\delta}| = \frac{\epsilon}{\delta} > \epsilon \text{ (Since } \ 0 < \delta < 1).$$

Thus, we have a contradiction and it follows that $f(x) = \frac{1}{x}$ cannot be uniformly continuous in $0 < x < 1$

1.6 Conclusion

I am convinced that we have justified our claim at the beginning of this unit when we said that continuous functions is one of the most important class of functions in analysis. This was a result to the rich harvest of theorems which have given us considerable depth in our discussion. Of practical importance is the Intermediate-Value theorem which requires deep understanding of continuity to be able to comprehend easily. We advised that students should practice most of the examples given and equally work through the theorems as well. In our next unit, we shall be considering sequences of functions.

1.7 Summary

We covered various theorems in this unit – one of the most important is the Intermediate-Value theorem. We equally discussed the boundedness of functions, continuous functions on a closed interval, and concluded with uniformly continuous functions.

1.8 Tutor Marked Assignments

1. Investigate the continuity of the following:

   (a) $f(x) = x^2$ at $x = 2$.

   (b) $f(x) = x\sin(\frac{1}{x})$ at $x = 0$.

2. Prove that $f(x) = \frac{x}{(x+2)}$ is continuous at the interval $1 < x < 3$.

3. Prove that $f(x) = x^2 - 4x + 3$ is uniformly continuous in the interval $2 \leq x \leq 4$.

4. Let $f_n x = nxe^{-nx^2}$, $n = 1, 2, 3, \ldots$, on $0 \leq x \leq 1$. Investigate the uniform convergence of the sequence $(f_n(x))$

1.9 References

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Unit 3: Sequences of Functions

1.1 Introduction

In the preceding two units, we considered the convergence of sequences of elements in R; in the present unit, we shall discuss sequences of functions. Thereafter, we shall introduce the basic notation of uniform convergence of a sequence of functions.

Unless there is a special mention to the contrary, we shall consider functions which have their common domain D in the Cartesian space R^p, and their range in R^q. We shall use the same symbols to denote the algebraic operations and the distances in the spaces R^p and R^q.

If for each natural number n there is a function f_n with domain D and range in R^q, we shall say that (f_n) is a sequence of functions on D to R. It should be noted that, for any point x in D such a sequence of functions gives a sequence of elements in R^q, namely, the sequence

$$(f_n(x)) \tag{1.1}$$

which is obtained by evaluating each of the functions at x. For certain points x in D the sequence (1.1) may converge and for other points x in D this sequence may diverge. For each of these points x for which the sequences (1.1) converges, there is a uniquely determined point of R^q. In general, the value of this limit, when it exists, will depend on the choice of the point x. In this way, there arises a function whose domain consists of all points x in $D \subseteq R^q$ for which the sequence (1.1) converges in R^q.

1.2 Objective

We have discussed sequence in R in our preceding units and since most problems in real life are captured in the form of functions, the study of sequence of functions, is therefore imperative in order for us to understand their behaviors with respect to their convergence. Just as we did in sequences of real numbers, at the end of this unit you should be able to establish the convergence or otherwise of sequences of functions and to determine the limit whenever it exists.

Definition 20 *Let (f_n) be a sequence of functions with common domain D in R^p, and with range in R^q, let D_0 be a subset of D, and let f be a function with domain containing D_0 and range in R^q. We say that the sequence (f_n) converges on D_0, to f if, for each x in D_0, the sequence $(f_n(x))$ converges in R^q to $f(x)$. When such a function f exists, we say that the sequence (f_n) converges to f on D_0, or simply that the sequence is convergent on D_0.*

It follows that , except for possible restrictions of the domain D_0, the limit function is uniquely determined. Ordinarily, we choose D_0 to be the largest set possible; that is, the set of all $x$in D for which (1.1) converges. In order to symbolize that the sequence (f_n) converges on D_0 to f, we sometimes write

$$f = \lim(f_n) \text{ on } D_0, \text{ or } f_n \to f \text{ on } D_0.$$

We shall now consider some examples of this idea. For simplicity, we shall treat the special case $p = q = 1$.

Examples 8.1

1. For each natural number n, let f_n be defined for x in $D = R$ by

   $$f_n(x) = \frac{x}{n}.$$

   Let f be defined for all x in $D = R$ by $f(x) = 0$. The statement that the sequence (f_n) converges on R to f is equivalent to the statement that for each real number x, the numerical sequence $(\frac{x}{n})$ converges to 0.

2. Let $D = \{x \in R : 0 \leq x \leq 1\}$ and for each natural number n, let f_n be defined by $f_n(x) = x^n$ for all $x \in D$ and let f be defined by

   $$\begin{aligned} f(x) &= 0, \; 0 \leq x \leq 1 \\ &= 1, \; x = 1 \end{aligned}$$

   It is clear that when $x = 1$, then

   $$f_n(x) = f_n(1) = 1^n = 1$$

   So that $f_n(1) \to f(1)$.

Observe that if $0 \le x \le 1$, then

$$f_n(x) = x^n \to 0$$

Therefore, we conclude that (f_n) converges on D to f. Equally, if $x > 1$, then $(f_n(x))$ does not converge at all.

3. Let $D = R$ and for each natural number n, let f_n be the functions defined for x in D by

$$f_n(x) = \frac{x^2 + nx}{n},$$

and let $f(x) = x$. Since

$$f_n(x) = \frac{x^2}{n} + x,$$

it follows that $(f_n(x))$ converges to $f(x)$ for all $x \in R$.

4. Let $D = R$ and for each natural number n, let f_n be defined to be

$$f_n(x) = \frac{1}{n}\sin(nx + n).$$

A rigorous definition of the sine function is not needed here; in fact, all we require is that

$$|\sin y| \le 1 \quad \text{for any real number } y.$$

If f is defined to be the zero function $f(x) = 0$, $x \in R$, then

$$f = \lim(f_n) \quad \text{on } R.$$

Indeed, for any real number x, we have

$$|f_n(x) - f(x)| = \frac{1}{n}|\sin(nx + n)| \le \frac{1}{n}.$$

If $\epsilon > 0$, there exists a natural number $n_0(\epsilon)$ such that if $n > n_0(\epsilon)$, then $\frac{1}{n} < \epsilon$. Hence for such n, we conclude that

$$|f_n(x) - f(x)| < \epsilon$$

no matter what the value of x. Therefore, w infer that the sequence (f_n) converges to f. Note that by choosing n sufficiently large, we can make the difference

$$|f_n(x) - f(x)|$$

arbitrarily small for all values of x simultaneously.

Partly to reinforce definition 20, and partly to prepare the way for the important notion of uniform convergence, we formulate the following restatement of definition 20.

Lemma 4 *A sequence (f_n) of functions on $D \subseteq R^q$ converges to a function f on a set $D_0 \subseteq D$ if and only if for each $\epsilon > 0$, and each x in D_0 there is a natural number $n_0(\epsilon, x)$ such that if $n \geq n_0(\epsilon)$, then*

$$|f_n(x) - f(x)| < \epsilon \tag{1.2}$$

Since this is just a reformulation of Definition 20, we shall not go through the details of the proof, but shall leave them to the student as exercises. We wish to only point out that the value of n required in (1.2) will depend, in general, on both $\epsilon > 0$ and $x \in D_0$. You will have already noted that, in Examples 8.1 (1-3), the value of n required to obtain (1.2) does depend on both $\epsilon > 0$ and $x \in D_0$ provided n is chosen sufficiently large but dependent on ϵ alone.

It is precisely this rather subtle difference which distinguishes between the notion of ordinary convergent of a sequence of functions (in the sense of Definition 20), and uniform convergence , which we now define.

Definition 21 *A sequence (f_n) of functions on $D \subseteq R^p$ to R^q converges uniformly on a subset D_0 of D to a function f in case for each $\epsilon > 0$, there is natural number $n_0(\epsilon)$ (depending on ϵ but not on $x \in D_0$) such that if $n > n_0(\epsilon)$, and $x \in D_0$, then*

$$|f_n(x) - f(x)| < \epsilon$$

In this case, we say that the sequence is *uniformly convergent* on D_0.

It follows immediately that if the sequence (f_n) is uniformly convergent on D_0 to f, then this sequence of functions also converges to f in the sense of Definition 20. That the converse is not true is seen by a careful examination of Examples 8.1 (1-3). Other examples will be given below. However, before we proceed, it is useful to state a necessary and sufficient condition for the sequence (f_n) to fail to converge uniformly on D_0 to f.

Lemma 5 *A sequence (f_n) does not converge uniformly on D_0 to f if and only if for some $\epsilon_0 > 0$, there is a subsequence (f_{nk}) of (f_n) and sequence (x_k) in D_0 such that*

$$|f_{nk}(x_k) - f(x_k)| \geq \epsilon_0 \quad \textit{for } k \in N$$

The proof of this result merely requires that the student negate Definition 21. It will be left to the student as exercise. The preceding Lemma is useful to show that Examples 8.1 (1-3) do not converge uniformly on the given sets D_0.

Examples 8.2

1. We consider Example 8.1(1). If $n_k = k$ and $x_k = k$, then $f_k(x_k) = 1$ so that

$$|f_k(x_k) - f(x_k)| = |1 - 0| = 1$$

This shows that the sequence (f_n) does not converge uniformly on R to f.

2. We consider Examples 8.1(2). If $n_k = k$ and $x_k = (\frac{1}{2})^{\frac{1}{k}}$, then

$$|f_k(x_k) - f(x_k)| = |f_k(x_k)| = \frac{1}{2}$$

Therefore, we infer that the sequence (f_n) does not converge uniformly on $\{x \in R : 0 \leq x \leq 1\}$ to f.

3. We consider Example 8.1(3). If $n_k = k$ and $x_k = k$, then

$$|f_k(x_k) - f(x_k)| = K,$$

Showing that (f_k) does not converge uniformly on R to f.

4. We consider Example 8.1(4). Then, since

$$|f_n(x) - f(x)| \leq \frac{1}{n} \;\; \forall\, x \in R,$$

the sequence (f_n) converges uniformly on R to f. However, if we restrict our attention to $D = [0, 1]$ and shuffle (f_n) with (g_n), whose $g_0(x) - x^n$, the resulting sequence (h_n) converges on D to the zero function. That the convergence of (h_n) is not uniform can be seen by looking at the subsequence $(g_n) = h_{2n}$ of h_n.

In order to establish uniform convergence, it is often convenient to make use of the notion of the norm of a function.

Definition 22 *If f is a bounded function defined on a subset D of R^p with values in R^q, the* D-norm *of f is the real number given by*

$$\|f\|_D = sup\{f(x) : x \in D\} \tag{1.3}$$

When the subset D is understood, we can safely omit the subscript on the left side of (8.3) and denote the D-norm of f by $\|f\|$.

Lemma 6 *If f and g are bounded functions defined on $D \subseteq R^p$ to R^q, then the D-norm satisfies the following relations:*

1. $\|f\| = 0$ if and only if $f(x) = \theta \;\; \forall\, x \in D$.

2. $\|cf\| = |c|\|f\|$ for any real number c.

3. $|\|f\| \pm \|g\|| \leq \|f \pm g\| \leq \|f\| \pm \|g\|$.

Proof.

1. If $f(x) = \theta \;\; \forall\, x \in D$, then

$$|f(x)| = |\theta| = 0 \;\; \forall\, x \in D$$

So that

$$\|f\| = sup\{|f(x)| : x \in D\} = 0.$$

Consequently, if there exists an element $x_0 \in D$ with $f(x_0) \neq \theta$, then $|f(x_0)| > 0$ and hence

$$\|f\| \leq |f(x_0)| > 0$$

2. This follows since $|cf(x)| = |c||f(x)|$.

3. According to triangle inequality,

$$|f(x) \pm g(x)| \leq |f(x)| \pm |g(x)|,$$

and by Definition 22, the RHS is dominated for each $x \in D$, by $\|f\| + \|g\|$. Therefore, the last number is an upper bound for the set

$$\{|f(x) \pm g(x)| : x \in D\}$$

So we conclude that

$$\|f \pm g\| \leq \|f\| \pm \|g\|.$$

Lemma 7 *A sequence (f_n) of bounded functions on $D \subseteq R^p$ to R^q converges uniformly on D to a function f if and only if*

$$\|f_n - f\| \to 0$$

Proof. If the sequence (f_n) converges to f uniformly on D, then for $\epsilon > 0$, there exists a natural number $n_0(\epsilon)$ and $x \in D$, then

$$|f_n(x) - f(x)| < \epsilon$$

This implies that if $n > n_0(\epsilon)$, then

$$\|f_n - f\| = sup\{|f_n(x) - f(x)| : x \in D\} \leq \epsilon$$

Hence $\|f_n - f\|$ converges to zero.

Conversely, if $\|f_n - f\|$ converges to zero, then $\epsilon > 0$ and $x \in D$, we have

$$|f_n(x) - f(x)| \leq \|f_n - f\| < \epsilon, \quad \text{provided that } n > n_0(\epsilon)$$

Therefore, if $x \in D$ and $n > n_0(\epsilon)$, then

$$|f_n(x)| - f(x)| < \epsilon$$

This shows that the sequence (f_n) converges uniformly on D to the function f.

Examples 8.3

1. We cannot apply Lemma 7 to the example considered Examples 8.1(1) and 8.2(1) for the reason that the functions f_n, defined to be $f_n(x) = \frac{x}{n}$, are not bounded on R, which was given as the domain. For the purpose of illustration, we change the domain to obtain a bounded sequence on the new domain. For convenience, let us take $D = [0, 1]$. Although the sequence $(\frac{x}{n})$ did not change uniformly to the zero function on the domain R (as was seen in Example 8.2(1)), the convergence is uniform on $D = [0, 1]$. To see this, we calculate the D-norm of $f_n - f$. In fact,

$$\|f_n - f\| = sup\{|\frac{\alpha}{n} - 0| : 0 \leq x \leq 1\} = \frac{1}{n},$$

and hence

$$\|f_n - f\| = \frac{1}{n} \to 0.$$

2. We now consider the sequence discussed in Examples 8.1(2) and 8.2(2) without changing the domain. Hence $D = \{x \in R : x > 0\}$ and $f_n(x) = x^n$.

   The set D_0 on which convergence takes place is $D_0 = [0, 1]$ and the limit function f is equal to 0 for $0 \leq x \leq 1$ and equal to 1 for $x = 1$. Calculating the $D_0 - -norm$ of the difference $f_n - f$, we have

$$\|f_n - f\| = sup \left\{ \begin{array}{ll} x^n, & 0 \leq x < 1 \\ 0, & x = 1 \end{array} \right\} = 1 \quad \text{for} \quad n \in N$$

   Since this $D_0 - -norm$ does not converge to zero, we infer that the sequence (f_n) does not converge uniformly on $D_0 = [o, 1]$ to f. This bears out our earlier considerations.

3. We consider Example 8.1(3). Once again, we cannot apply Lemma 7, since the functions are not bounded. Again, we choose a smaller domain, taking $D_0 = [0, a]$ with $a > 0$. Since

$$|f_n(x) - f(x)| = |\frac{x^2 + nx}{n} - x| = \frac{x^2}{n},$$

   the $D - -norm$ of $f_n - f$ is

$$\|f_n - f\| = sup\{|f_n(x) - f(x)| : 0 \leq x \leq a\} = \frac{a^2}{n}.$$

   Hence the sequence converges uniformly to f on the interval $[0, a]$.

4. Referring to Example 8.2(4), we consider the function

$$f_n(x) = \frac{1}{n} \sin(nx + n) \quad \text{on} \quad D = R.$$

   Hence, the limit function $f(x) = 0$ for all $x \in D$. In order to establish the uniform convergence of this sequence, note that

$$\|f_n - f\| = sup\{(\frac{1}{n})| \sin(nx + n) : x \in R\}.$$

   But, since $|\sin g| \leq 1$, we conclude that

$$\|f_n - f\| = \frac{1}{n}.$$

   Hence, (f_n) converges uniformly on R, as was established in Example 8.2(4).

   One of the more useful aspects of the norm is that it facilitates the formulation of a Cauchy Criterion for the uniform convergence of a sequence of bounded functions.

1.2.1 Cauchy Criterion for Uniform Convergence

Let (f_n) be a sequence of bounded functions, on D in R^p with values in R^q. Then there is a function to which (f_n) is uniformly convergent on D if and only if for each $\epsilon > 0$, there is a natural number $M(\epsilon)$ such that if $m, n \leq M(\epsilon)$, the D–norm satisfies

$$\|f_m - f_n\| < \epsilon.$$

Proof. Suppose the sequence (f_n) converges uniformly on D to a function f. Then, for $\epsilon > 0$, there is a natural number $n_0(\epsilon)$ such that if $n > n_0(\epsilon)$, then the D-norm satisfies

$$\|f_n - f\| < \frac{\epsilon}{2}$$

Hence, if both $m, n > n_0(\epsilon)$, we conclude that

$$\|f_m - f_n\| \leq \|f_m - f\| + \|f - f_n\| < \epsilon.$$

Conversely, suppose the Cauchy Criterion is satisfied and that for $\epsilon > 0$, there is a natural number $M(\epsilon)$ such that the D–norm satisfies

$$\|f_m - f_n\| < \epsilon \quad \text{whenever} \quad m, n > M(\epsilon).$$

Now, for each $x \in D$, we have

$$|f_m(x) - f_n(x)| \leq \|f_m - f_n\| < \epsilon \quad \text{for} \ \ m, n > M(\epsilon).$$

Hence, the sequence $(f_n(x))$ is a Cauchy sequence in R^q and so converges to some element of R^q. we define f for x in D by

$$f(x) = \lim(f_n(x)).$$

We conclude that if m is a fixed natural number satisfying $m > M(\epsilon)$, and if n is any natural number with $n > M(\epsilon)$, then for all x in D, we have

$$|f_m(x) - f_n(x)| < \epsilon$$

It follows that for $m > M(\epsilon)$ and $x \in D$, then

$$|f_m(x) - f_n(x)| \leq \epsilon.$$

Therefore, the sequence (f_m) converges uniformly on E to the function f.

1.3 Conclusion

The ground for this unit was prepared by the two subsequent units in this Module, instead of the sequence of numbers in R, we now replaced these numbers with functions. If you observe carefully, you would find some similarities that will make your reading of this unit quite interesting. Towards the end of the unit, we introduced D-norm concept that enables us to discuss uniform continuous functions. The examples were quite elaborate to capture every behaviors exhibited by the sequence of functions. This marks the end of this Module, our next Module will present sequences of series.

1.4 Summary

This unit was essentially about the convergence of sequences of functions. The conditions for the sequence of functions to converge uniformly was discussed employing the concept of D-norm. Each of the property of the sequence of functions were illustrated with appropriate examples.

1.5 Tutor Marked Assignments

1. Let

$$f_n(x) = nxe^{-nx^2}, \quad n = 1, 2, 3, \dots \;\; 0 \leq x \leq 1.$$

   Investigate the uniform convergence of the sequence $(f_n(x))$.

2. For each $n \in N$, let f_n be defined for $x > 0$ by

$$f_n(x) = (\frac{1}{nx}).$$

   For what values of x does $\lim(f_n(x))$ exist?

3. Show that, if we define f_n on R by

$$f_n(x) = \frac{nx}{1 + n^2x^2}$$

   then f_n converges on R.

1.6 References

[1.] Anderson, K,W., and D.W. Hull, *Sets, Sequences, and Mappings*, John Wiley, New York, 1963.

[2.] Graves, L.M., *Theory of Foundations of Real Variables*, Second Edition, McGraw-Hill, New York, 1968.

[3.] Krantz, S., *Real Analysis and Foundations*, CRC Press, 1991.

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PART 4: SERIES

Unit 1: Convergence of Series

1.1 Introduction

This Module as a whole to be covered in three units is concerned with establishing important theorems in the theory of infinite series. In this first unit, we shall present the main theorems concerning the convergence of infinite series in R. We shall obtain some results of a general nature which serve to establish the convergence of the series and justify certain manipulation with series.

1.2 Objective

This unit will enable us to study the behavior of sequences which are obtained by attaching sums to the given sequence which in other words generates the infinite series which is the subject of this unit. It will enable us to consider the behavior of some special series such as geometric series, harmonic series, and p-series which are common in most mathematical methods. At the end of this unit, we should be able to relate our understanding of convergence of sequences to the convergence of series, and be able to establish the convergence of series.

1.3 Infinite Series

In elementary mathematics, an infinite series is sometimes defined to be an expression of the form

$$x_1 + x_2 + \cdots + x_n + \cdots$$

This definition lacks clarity, however, since there is no particular value that we can attach a priori to this array of symbols which calls for an infinite number of additions to be performed. Although, there are other definitions that are suitable, we shall take an infinite series to be the same as the sequence of partial sums.

Definition 23 *If $X = (x_n)$ is a sequence in R, then the infinite series (or simply the series) generated by X is the sequence $S = (s_k)$ defined by*

$$\begin{aligned} s_1 &= x_1, \\ s_2 &= s_1 + x_2 (= x_1 + x_2), \\ \cdots & \cdots \cdots\cdots\cdots \\ s_k &= s_{k-1} + x_k (= x_1 + x_2 + \cdots + x_k). \end{aligned}$$

If S converges, we refer to $\lim S$ as the sum of the infinite series. The elements x_n are called the *terms* and the elements s_k are called the *partial sums* of this infinite series.

It is conventional to use the expression above or one of the symbols given below,

$$\sum (x_n), \quad \sum_{n=1}^{\infty} (x_n), \quad \sum_{n=1}^{\infty} x_n$$

both to denote the infinite series generated by the sequence $X = (x_n)$ and also to denote $\lim S$ in the case that this limit is convergent.

The student should guard against confusing the words "sequence" and "series". In non-mathematical language, these words are inter-changeable, in mathematics, however, they are not synonyms. According to our definition, an infinite series is a sequence S obtained from a given sequence X according to a special procedure that was stated above. There are many other ways of generating new sequences and attaching "sums" to the given sequence X.

Although we generally index the elements of the series by natural numbers, it is sometimes more convenient to start with $n = 0$, with $n = 4$, or with $n = k$. When such is the case, we shall denote the resulting series or their sums by notations such as

$$\sum_{n=0}^{\infty} x_n, \quad \sum_{n=4}^{\infty} x_n \quad \text{or} \quad \sum_{n=k}^{\infty} x_n$$

In the elementary properties of a sequence given in Module 2, Unit 1, we defined the sum and difference of two sequences X,Y in R. Similarly, if c is a real number and if w defines the sequences $cX = (cx_n)$ in R. We now examine the series generated by these sequences.

Let us start from an example which is likely to be familiar from elementary mathematics. Let us look at the infinite sum

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$$

This is known as the sum of a geometric progression. One proceeds as follows, define the sum of the 1st n terms by

$$s_n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} = 1 - \frac{1}{2^n}$$

and then define the infinite sum S as $\lim_{n\to\infty} s_n$, so that $s = 1$.

This idea is used to define an arbitrary series.

Definition 24 *Let (x_n) be a sequence of real numbers, suppose*

$$s_n = x_1 + x_2 + \cdots + x_n = \sum_{k=1}^{n} x_k$$

We say that the series

$$\sum_{k=1}^{\infty} x_k = x_1 + x_2 + \cdots$$

is convergent if the sequence of the partial sums s_n is convergent. The limit of this sequence is called the sum of the series.

If this series is not convergent, we say that it is divergent (or diverges).

Theorem 17 *If the series*

$$\sum_{n=1}^{\infty} x_n$$

is convergent, then $\lim_{n\to\infty} x_n = 0$.

Proof. Let

$$s_n = \sum_{k=1}^{n} x_k.$$

Then by the definition, the limit $\lim_{n\to\infty} s_n$ exists. Denote it by S. Then of course,

$$\lim_{n\to\infty} s_{n-1} = S$$

Note that $x_n = s_n - s_{n-1}$ for $n \geq 2$. Hence

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1} = S - S = 0$$

The same idea can be used to prove the following theorem.

Theorem 18 *If the series*

$$\sum_{n=1}^{\infty} x_n$$

is convergent, then

$$s_{2n} - s_n = x_{n+1} + x_{n+2} + \cdots + x_n \to 0 \text{ as } n \to \infty.$$

The above theorem expresses the simplest necessary condition for the convergence of a series.

For example, each of the following series is divergent,

$$1 + 1 + 1 + 1 + \quad \cdots$$

$$1 - 1 + 1 - 1 + \quad \cdots$$

Since the necessary condition is not satisfied.

Theorem 19 1. If the series $\sum(x_n)$ and $\sum(y_n)$ converge, then the series $\sum(x_n + y_n)$ converges and the sums are related by the formula

$$\sum(x_n + y_n) = \sum(x_n) + \sum(y_n).$$

A similar result holds for the series generated by $X - Y$.

2. If the series $\sum(x_n)$ is convergent, c is a real number, then the series $\sum(cx_n)$ converges, and

$$\sum(cx_n) = c\sum(x_n).$$

1.3.1 Cauchy Criterion for Series

The series $\sum(x_n)$ in R converges if and only if for each positive number ϵ, there is a natural number $n_0(\epsilon)$ such that if $m \geq n \geq n_0(\epsilon)$, then

$$|s_m - s_n| = |x_{n+1} + x_{n+2} + \cdots + x_m| < \epsilon$$

The statement may refer to the theory of Cauchy sequences covered in Module 2, Unit 2 if any difficulty is posed by this proposition.

1.4 Absolute Convergence

The notion of absolute convergence is often of great importance in treating series, as we shall show later.

Definition 25 *Let $X = (x_n)$ be a sequence in R. We say that the series $\sum(x_n)$ is* absolutely convergent *if the series $\sum|x_n|$ is convergent but not absolutely convergent.*

It is stressed that for series whose elements are non-negative real numbers, there is no distinction between ordinary convergence and absolute convergence. However, for other series there is a difference.

Theorem 20 *If a series in R is absolutely convergent, then it is convergent.*

Proof. By hypothesis, the series $\sum |x_n|$ converges. Therefore, it follows from the necessity of the Cauchy criterion above that given $\epsilon > 0$, there exists a natural number $n_0(\epsilon)$ such that if $m \geq n \geq n_0(\epsilon)$, then

$$|x_{n+1}| + |xn + 2| + \cdots + |x_m| < \epsilon$$

Thus, the L.H.S. of this relation dominates

$$|x_{n+1} + x_{n+2} + \cdots + x_m|.$$

We apply the sufficiency of the Cauchy criterion to conclude that the $\sum(x_n)$ must converge.

Examples

1. **Geometric Series:** We consider the real sequence $X = (a^n)$, which generates the geometric series

$$a + a^2 + \cdots + a^n + \cdots$$

A necessary condition for convergence is that $\lim_{n\to\infty}(a^n = 0)$, which requires that $|a| < 1$. If $m \geq n$, then

$$a^{n+1} + a^{n+2} + \cdots + a^m = \frac{a^{n+1} - a^{m+1}}{1-a},$$

as can be verified by multiplying both sides by $1 - a$ and noticing the telescoping on the left side. Hence the partial sums satisfy

$$|s_m - s_n| = |a^{n+1} + \cdots + a^m| \leq \frac{|a^{n+1}| + |a^{m+1}|}{|1-a|}; \;\; m \geq n.$$

If $|a| < 1$, then $|a^{n+1}| \to 0$ so the Cauchy criterion implies that the geometric series converges if and only if $|a| < 1$. Letting $n = 0$ in 9.3.1, and passing to the limit with respect to m, we find that it converges to the limit

$$\frac{a}{1-a}, \quad \text{when} \quad |a| < 1$$

2. **Harmonic Series:** Consider the harmonic series $\sum(\frac{1}{n})$, which is well known to diverge. Since $\lim \frac{1}{n} = 0$. We need to show that a subsequence of the partial sums in not bounded. In fact, if $k_1 = 2$, then

$$s_{k_1} = \frac{1}{1} + \frac{1}{2};$$

and if $k_2 = 2^2$, then

$$s_{k_1} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = s_{k_1} + \frac{1}{3} + \frac{1}{4} > s_{k_1} + 2(\frac{1}{4}) = 1 + \frac{2}{2}$$

by mathematical induction, we establish that if $k_r = 2^r$, then

$$s_{k_r} > s_{k_{r-1}} + 2^{r-1}(\frac{1}{2^r}) = s_{k_{r-1}} + \frac{1}{2} = 1 + \frac{r}{2}$$

Therefore, the subsequence (s_{k_r}) is not bounded and the harmonic series does not converge.

3. **Power Series:** We now treat the *p-series* $\sum(\frac{1}{n^p})$ where $0 < p < 1$, and use the elementary inequality $n^p \leq n$, for $n \in N$. From this, it follows that, when $0 < p < 1$, the

$$\frac{1}{n} \leq \frac{1}{n^p}, \quad n \in N.$$

Since the partial sums of the harmonic series are not bounded, this inequality shows that the partial sums of

$$\sum \frac{1}{n^p}$$

are not bounded for $0 < p \leq 1$. Hence the series diverges for these values of p.

3. Consider the p-series for $p > 1$. Since the partial sums are monotonic, it is sufficient to show that some subsequence remains bounded in order to establish the convergence of the series. If $k_1 = 2^1 - 1 = 1$, then $s_{k_1} = 1$. If $k_2 = 2^2 - 1 = 3$, we have

$$s_{k_2} = \frac{1}{1} + (\frac{1}{2^p} + \frac{2}{3^p}) < 1 + \frac{2}{2^p} = 1 + \frac{1}{2^{p-1}},$$

and if $k_3 = 2^3 - 1$, we have

$$s_{k_2} + s_{k_2} + (\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{7^p}) < s_{k_2} + \frac{4}{4^p} < 1 + \frac{1}{2^{p-1}} + \frac{1}{4^{p-1}},$$

Let $a = \frac{1}{2^{p-1}}$; since $p > 1$, it is seen that $0 < a < 1$. By mathematical induction, we find that if $k_r = 2^r - 1$, then

$$0 < s_{k_r} < 1 + a + a^2 + \cdots + a^{r-1}.$$

Hence the number $\frac{1}{1-a}$ is an upper bound for the partial sums of the p-series when $1 < p$. It therefore follows that for such values of p , the p-series converges.

5. Consider the series $\sum(\frac{1}{n^2+n})$. By using partial fractions, we can write

$$\frac{1}{k^2+k} = \frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$$

This expression shows that the partial sums are telescoping and hence

$$s_n = \frac{1}{1.2} + \frac{1}{2.3} + \cdots + \frac{1}{n(n+1)} = \frac{1}{1} - \frac{1}{n+1}$$

It follows that the sequence (s_n) is convergent to 1.

1.5 Conclusion

We have been able to establish very important theorems in the theory of infinite series. We equally draw the distinction between convergence of sequences and the convergence of series. According to our definition, our infinite series is a sequence S obtained from a given sequence X according to a special procedure. Because of the importance of series in mathematics generally, we have taken time to present elaborate examples to provide an insight into the special nature of convergent series. In the next unit, we shall be discussing the various tests for convergence of series to be able to establish the necessary and sufficient condition for convergence of infinite series.

1.6 Summary

This unit centered on the convergence of series. Various theorems were presented to give adequate clarity to the convergence of series and concluded with the concept of absolute convergence.

1.7 Tutor Marked Assignments

1. Prove that

$$\sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$$

is uniformly convergent for all x.

2. If

$$\sum_{n=0}^{\infty} a_n x^n$$

converges for $x = x_0$, prove that it converges uniformly and absolutely in the interval

$$|x| < |x_1| \quad \text{where} \quad |x_1| < |x_0|.$$

1.8 References

[1.] Graves, L.M., *Theory of Foundations of Real Variables*, Second Edition, McGraw-Hill, New York, 1968.

[2.] Knopp, K., *Theory and Application of Infinite Series*, R.C. Young, translator, Hafner, New York, 1951

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Unit 2: Tests for Convergence

1.1 Introduction

In the preceding unit,we obtained some results concerning the manipulation of infinite series, especially in the important case where the series are absolutely convergent. However, except for the Cauchy criterion, and the fact that the terms of a convergent series converge to zero, we did not establish any necessary or sufficient condition for convergence of infinite series.

We shall now give some results which can be used to establish the convergence or divergence of infinite series. In view of its importance, we shall pay special attention to absolute convergence. Since the absolute convergence of the series $\sum(x_n)$ in R is equivalent to the convergent of the series $\sum|x_n|$ of non-negative elements of R, it is clear that results establishing the convergence of non-negative real series have particular interest.

1.2 Objective

Our aim here is to discuss how to establish necessary and sufficient conditions for the convergence or divergence of infinite series and give some usefulness of various tests in mathematical analysis. For instance, one of the tests, the root test can be used to obtain an estimate of the rapidity of convergence which is useful in numerical computations and some theoretical estimates as well.

1.3 Comparison Test

Our first test, *comparison test* shows that if the terms of a non-negative real series are dominated by the corresponding terms of a convergent series, then the first series is convergent.

Definition 26 *Let $X = (x_n)$ and $Y = (y_n)$ be non-negative real sequences and suppose that for some natural number K,*

$$x_n \leq y_n \quad \text{for} \quad n \geq K.$$

Then the convergence of $\sum(x_n)$ implies the convergence of $\sum(x_n)$.

Proof. If $m \geq n \geq sup\{K, M(\epsilon)\}$, then

$$x_{n+1} + \cdots + \leq yn + 1 + \cdots y_m < \epsilon,$$

from which the assertion is evident.

1.3.1 Limit Comparison Test

Suppose, that $X = (x_n)$ and $Y = (y_n)$ are non-negative real sequences, then the following relations hold:

1. If the relation
$$\lim \left(\frac{x_n}{y_n}\right) \neq 0$$
holds, then $\sum(x_n)$ is convergent if and only if $\sum(y_n)$ is convergent.

2. If the limit in (1.) above is non-zero, and (y_n) is convergent, then $\sum(x_n)$ is convergent.

   Proof. Here, we want to establish tests which make possible to infer the convergence or divergence of a series by means of comparing it with another series convergence or divergence of which is known.

   It follows that for some real number $c > 1$, and some natural number, K, then
$$\left(\frac{1}{c}\right) y_n \leq x_n \leq c y_n, \qquad \text{for} \quad n \geq K$$

   If we apply the Comparison Test twice, we obtain the assertion in (2.) above. The details of the proof of (2.) are similar and will be left as exercise.

We now give an important test due to Cauchy.

1.4 Root Test

1. If $X = (x_n)$ is a sequence in R, and there exists a non-negative number $r < 1$, and a natural number K, such that

$$|x_n|^{\frac{1}{n}} \leq r \qquad n \geq K,$$

then the series $\sum(x_n)$ is absolutely convergent.

2. If there exists a real number $r > 1$, and a natural number K such that

$$|x_n|^{\frac{1}{n}} \geq r \qquad n \geq K,$$

then the series $\sum(x_n)$ is convergent.

Proof.

1. If 10.4 holds, then we have $|x_n| \leq r^n$. Now for $0 \leq r \leq 1$, the series $\sum(r^n)$ is convergent. Hence, it follows from the comparison test that $\sum(x_n)$ is absolutely convergent.

2. If (1.) holds, then $|x_n| \geq r^n$. However, since $r \geq 1$, it is false that $\lim(|x_n|) = 0$.

In addition to establishing the convergence of $\sum(x_n)$, the root test can be used to obtain an estimate of the rapidity of convergence. This estimate is useful in numerical computations and in some theoretical estimates as well.

Corollary 3 *If r satisfies $0 < r < 1$ and if the sequence $X = (x_n)$ satisfies the root test, then the partial sums s_n, $n \geq K$, approximate the sum $S = \sum(x_n)$ according to the estimate*

$$|s - s_n| \leq \frac{r^{n+1}}{1-r} \qquad \text{for } \ n \geq K.$$

Proof. If $m \geq n \geq K$, we have

$$|s_m - s_n| = |x_{n+1} + \cdots + x_m| \leq |x_{n+1}| + \cdots + |x_m| \leq r^{n+1} + \cdots + r^m < \frac{r^{n+1}}{1-r}$$

Now take the limit with respect to m to obtain the relation above.

It is often convenient to make use of the following variant of the root test.

Corollary 4 *Let $X = (x_n)$ be a convergence in R, and set*

$$r = \lim \left(|x_n|^{\frac{1}{n}} \right),$$

whenever this limit exists. Then $\sum(x_n)$ is absolutely convergent when $r < 1$ and is divergent when $r > 1$.

Proof. It follows that the limit exists and is less than 1, then there is a real number r_1 with $r < r_1 < 1$ and a natural number K such that

$$|x_n|^{\frac{1}{n}} \leq r_1, \qquad \text{for } \ n \geq K.$$

In this case, the series is absolutely convergent. If this limit exceeds 1, then there is a real number $r_2 > 1$ and a natural number K such that

$$|x_n|^{\frac{1}{n}} \geq r_2, \qquad \text{for } \ n \geq K.$$

in which case the series is divergent.

1.5 Ratio Test (D'Alembert)

1. If $X = (x_n)$ is a sequence of non-zero elements of R, and there is a positive number $r < 1$ and a natural number K such that

$$\frac{|x_{n+1}|}{|x_n|} \leq r, \qquad \text{for } \ n \geq K$$

then the series $\sum(x_n)$ is absolutely convergent.

2. If there exists a number $r > 1$ and a natural number K such that,

$$\frac{|x_{n+1}|}{|x_n|} \geq r, \qquad \text{for } \ n \geq K$$

then the series $\sum(x_n)$ is divergent.

Proof.

1. If (1.) above holds, then an induction argument shows that

$$|x_{k+m}| \leq r^m |x_k|, \qquad \text{for } \ m \geq 1$$

It follows that for $n \geq K$, the terms of $\sum(x_n)$ dominated by a fixed multiple of the terms of the geometric series $\sum(r^n)$ with $0 \leq r < 1$. From the comparison test, we infer that $\sum(x_n)$is absolutely convergent.

2. If (2.) above holds, then an induction argument shows that

$$|x_{k+m}| \geq r^m |x_k|, \qquad \text{for } \ m \geq 1$$

Since $r \geq 1$, it is impossible to have $\lim(|x_n|) = 0$, so the series cannot converge.

Corollary 5 *If r satisfies $0 \leq r < 1$, and if the sequence $X = (x_n)$ satisfies the ratio test above for $n \geq K$, then the partial sums approximate the sum $S = \sum(x_n)$ according to the estimate,*

$$|s - s_n| \leq \frac{r}{1-r}|x_n|, \quad \text{for } \ n \geq K.$$

Proof. The above relation implies that

$$|x_{n+k}| \leq r^k |x_n|, \quad \text{when } \ n \geq K.$$

Therefore, if $m \geq n \geq K$, we have

$$\begin{aligned} |s_m - s_n| = |x_{n+1} + \cdots + x_m| &\leq |x_{n+1}| + \cdots + |x_m| \\ &\leq (r + r^2 + \cdots + r^{m-n})|x_m| \\ &< \tfrac{r}{1-r}\,|x_n|\,. \end{aligned}$$

Corollary 6 *Let $X = (x_n)$ be a sequence in R and set*

$$r = \lim\left(\frac{|x_{n+1}|}{|x_n|}\right),$$

whenever the limit exists. Then the series $\sum(x_n)$ is absolutely convergent when $r < 1$, and divergent when $r > 1$.

Proof. Suppose that the limit exists and $r < 1$. If r_1 satisfies $r < r_1 < 1$, then there exists a natural number K such that

$$\frac{|x_{n+1}|}{|x_n|} > r_2, \quad \text{for } \ n \geq K,$$

and in this case, there is divergence.

Examples

1. First we shall apply the comparison test. Knowing that the harmonic series $\sum(\frac{1}{n})$ diverges, it is seen that if $p \leq 1$, the $n^p \leq n$ and hence

$$\frac{1}{n} \leq \frac{1}{n^p}.$$

After using the comparison test, we conclude that the p-series $\sum(\frac{1}{n^p})$ diverges for $p \leq 1$.

2. Now consider the case $p = 2$, that is the series $\sum(\frac{1}{n^2})$. We compare the series with the convergent series $\sum(\frac{1}{n(n+1)})$ of Examples 9.1(5.) in Unit 1. Since the relation

$$\frac{1}{n(n+1)} < \frac{1}{n^2}$$

holds and the terms on the left form a convergent series, we cannot apply the comparison theorem directly. However, we could apply this theorem if we compared the nth term of

$$\sum\left(\frac{1}{n(n+1)}\right)$$

with the $(n+1)st$ term of $\sum(f1n^2)$. Instead, we choose to apply the limit comparison test and note that

$$\frac{1}{n(n+1)} \div \frac{1}{n^2} = \frac{n^2}{n(n+1)} = \frac{n}{n+1}.$$

Since the limit of this quotient is 1, and $\sum(\frac{1}{n(n+1)})$ converges, then so does the series $\sum(\frac{1}{n^2})$.

3. Now consider the $p \geq 2$. If we note that

$$n^p \geq n^2, \quad \text{for } p \geq 2$$

then

$$\frac{1}{n^p} \leq \frac{1}{n^2}$$

and direct application of the comparison test assures that $\sum(\frac{1}{n^p})$ converges for $p \leq 2$. Alternatively, we could apply the limit comparison test and note that

$$\frac{1}{n^p} \div \frac{1}{n^2} = \frac{n^2}{n^p} = \frac{1}{n^{p-2}}.$$

If $p > 2$, this extension converges to 0, when it follows from 10.3.1(2) that the series $\sum(\frac{1}{n^p})$ converges for $p \geq 2$.

By using the comparison test, we cannot gain any information concerning the p-series for $1 < p < 2$. Unless we can find a series whose convergence character is known and which can be compared to the series in this range.

4. We demonstrate the root and ratio tests as applied to the p-series. Note that

$$\left(\frac{1}{n^p}\right)^{\frac{1}{n}} = (n-p)^{\frac{1}{n}} = \left(n^{\frac{1}{n}}\right)^{-p}.$$

Now it is known that the sequence $n^{\frac{1}{n}}$ converges to 1. Hence we have

$$\lim \left(\left(\frac{1}{n^p} \right)^{\frac{1}{n}} \right) = 1$$

so that the root test does no apply.

Similarly, since

$$\frac{1}{(n+1)^p} \div \frac{1}{n^p} = \frac{n^p}{(n+1)^p} = \frac{1}{(1+\frac{1}{n})^p},$$

and since the sequence $(1+\frac{1}{n})^p$ converges to 1, the ratio test does not apply.

1.6 Conclusion

At the beginning of this unit, we set out to provide ways in which we can establish the necessary and sufficient conditions for the convergence or otherwise of an infinite series, and we hope we have been able to justify our aim. We equally presented elaborate theorems, corollary's and concise examples that demonstrates our thought. Our next and concluding unit shall be concerned with Power Series and Uniform Convergence.

1.7 Summary

We covered three major tests for the convergence of series in this unit. These are Comparison test, Root test, and Ratio test. We presented theorems, corollary's and examples to illustrate our thought elaborately.

1.8 Tutor Marked Assignments

1. Investigate the convergence of the series

$$\sum_{n=1}^{\infty} \frac{2^n + n}{3^n - n}.$$

2. Investigate the convergence of the series

$$\sum_{n=1}^{\infty} 2^{(-1)^n} \left(\frac{1}{2} \right)^n$$

1.9 References

[1.] Graves, L.M., *Theory of Foundations of Real Variables*, Second Edition, McGraw-Hill, New York, 1968.

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Unit 3: Uniform Convergence and Power Series

1.1 Introduction

Because of their frequent appearances and importance, we shall conclude this Module with a discussion of infinite series of functions. Since the convergence of an infinite series is handled by examining the sequence of partial sums, questions concerning the series of functions are answered by examining corresponding questions for sequences of functions. For this reason, a portion of the present unit is merely a translation of facts already established for sequences of functions into series terminology. However, in the second part of the unit, we shall discuss power series, where some new features arise merely because of the special character of the functions involved.

1.2 Objective

One of the main reasons for the interest in uniformly convergent series of functions is the validity continuous functions and the integrability of real-valued functions in integral theory. While power series is an important class of series of functions because it enjoys properties that are not valid for general series of functions.

1.3 Series of Functions

Definition 27 *If (f_n) is a sequence of functions defined on a subset D of R^p with values in R^q, the sequence of partial sums (s_n) of the infinite series, $\sum(f_n)$ is defined for x in D by*

$$\begin{aligned} s_1(x) &= f_1(x), \\ s_2(x) &= s_1(x) + f_2(x) \quad [= f_1(x) + f_2(x)], \\ \cdots & \cdots \ldots\ldots\ldots\ldots\ldots\ldots \\ s_{n+1}(x) &= s_n(x) + f_{n+1}(x) \quad [= f_1(x) + f_2(x)], \\ \cdots & \cdots \ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

In case the sequence (s_n) converges on D to a function f, we say that the infinite series of functions, $\sum(f_n)$ converges to f on D. We shall often write

$$\sum(f_n), \qquad \sum_{n=1}^{\infty}(f_n), \quad \text{or} \quad \sum_{n=1}^{\infty} f_n$$

to denote either the series or the limit functions when it exists.

If the series $\sum(|f_n(x)|)$ converges for each x in D, then we say that $\sum(f_n)$ is absolutely convergent on D. If the sequence (s_n) is uniformly convergent on D to f, the we say that $\sum(f_n)$ is *uniformly convergent* on D, or that it converges uniformly on D.

Definition 28 *Let $\sum_{k=1}^{\infty}$ be a series of functions and*

$$s_n(x) = f_1(x) + f_2(x) + \cdots + f_n(x)$$

be the nth partial sum of the series. Suppose that some set A, for example $[a, b]$, the series converges to the sum $s(x)$. We say that the series converges uniformly to $s(x)$ in $[a, b]$ if given $\epsilon > 0$, there exists a positive integer n_0 such that

$$|s_n(x) - s(x)| < \epsilon \quad \forall\ n > n_0 \ \text{ and all } \ x \in [a, b].$$

Theorem 21 *If the functions $f_n(x)$ are continuous in a set A and if $\sum f_n(x)$ converges uniformly to $s(x)$ in A, then $s(x)$ is continuous in A.*

Proof. We prove the theorem for the case where $A = [a, b]$.

First we observe that if $s_n(x)$is the nth partial sum of the series and $r_n(x)$ is the remainder of the series after n terms, then

$$s(x) = s_n(x) + r_n(x)$$

$$\text{i.e.,} \qquad s(x+h) = s_n(x+x) + r_n(x+h)$$

$$\text{and} \qquad s(x+h) - s(x) = s_n(x+h) - s_n(x) + r_n(x+h) - r_n(x) \tag{1.4}$$

where we choose h so that both x and $x+h$ are in $[a,b]$ (if $x=b$, for example, this will require $h<0$).

Since $s_n(x)$ is the sum of the finite number of continuous functions, it must also be continuous. Then given $\epsilon > 0$, we can find δ so that

$$|s_n(x+h) - s_n(x)| < \frac{\epsilon}{3} \quad \text{whenever} \quad |h| < \delta \tag{1.5}$$

since the series, by hypothesis is uniformly convergent, we can choose $n_0(\epsilon)$, but not on x so that

$$r_n(x) < \frac{\epsilon}{3} \quad \text{and} \quad r_n(x+h) < \frac{\epsilon}{3} \quad \text{for} \quad n > n_0 \tag{1.6}$$

Then from (11.1), (11.2) and (11.3), we have

$$|s(x+h) - s(x)| \le |s_n(x+h) - s_n(x)| + |r_n(x+h)| + |r_n(x)| < \epsilon \quad \text{for} \quad |h| < \delta,$$

and so the continuity is established.

Theorem 22 Weierstrass M-Test *If* $|f_n(x)| \le M_n, \quad n = 1,2,3,\ldots$ *where* M_n *are positive constants such that* $\sum M_n$ *converges, then* $\sum f_n(x)$ *is uniformly and absolutely convergent.*

Proof. The remainder of the series $\sum f_n(x)$ after n terms is $r_n(x) = f_{n+1}(x) + f_{n+2} + \cdots + \cdots$. Now

$$\begin{aligned} |r_n(x)| = |f_{n+1}(x) + f_{n+2}(x) + \cdots| &\le |f_{n+1}(x)| + |f_{n+2}(x)| + \cdots \\ &\le M_{n+1} + M_{n+2} + \cdots \end{aligned}$$

But $M_{n+1} + M_{n+2} + \cdots$ can be made less than ϵ by choosing $n > n_0$ since $\sum M_n$ converges. Since n_0 is clearly independent of x, we have

$$|r_n(x)| < \epsilon \qquad \text{for} \quad n > n_0$$

and the series is uniformly convergent.

The absolute convergence follows from the fact that

$$\sum_{k=n+1}^{\infty} |f_k(x)| < \sum_{k=n+1}^{\infty} M_k < \epsilon \quad \text{for} \quad n > n_0.$$

Examples 11.1

1. Prove that
$$\sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$$
is uniformly convergent for all x.

   We have for all x,
$$\left|\frac{\cos nx}{n^2}\right| \leq \frac{1}{n^2}$$
   Then since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, the series is uniformly convergent by the Weierstrass M-Test of Theorem 22.

2. Consider the series
$$\sum_{n=1}^{\infty} \frac{x^2}{n^2}.$$
   If $|x| \leq 1$, then
$$\left|\frac{x^n}{n^2}\right| \leq \frac{1}{n^2}$$
   Since the series $\sum(\frac{1}{n^2})$ is convergent, it follows from Weierstrass M-test that the given series is uniformly convergent on the interval $[-1, 1]$.

1.4 Power Series

We shall now turn our discussion to power series. This is important class of series of functions and enjoys properties that are not valid of general series of functions.

Definition 29 *A series of real functions* $\sum(f_n)$ *is said to be a* power series *around* $x = c$ *if the function* f_n *has the form*
$$f_n(x) = a_n(x-c)^n,$$
where a_n and c belong to R and where $n = 0, 1, 2, \ldots$.

For the sake of simplicity of our notation, we shall treat only the case where $c = 0$. This is no loss of generality, however, since the translation $x' = x - c$ reduces a power series around c to a power series around 0. Thus, whenever we refer to a power series, we shall mean a series of the form
$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + \cdots + a_n x^n + \cdots \tag{1.7}$$

Even though the functions appearing above are defined over all of R, it is not to be expected that the series (11.4) will converge for all x in R. For example, by using the Ratio test , we can show that the series

$$\sum_{n=0}^{\infty} n!x^n, \qquad \sum_{n=0}^{\infty} x^n, \qquad \sum_{n=0}^{\infty} \frac{x^n}{n!},$$

converge for x in the sets

$$\{0\}, \{x \in R : |x| < 1\}, R,$$

respectively. Thus, the set on which a power series converges may be small, medium, or large. However, an arbitrary subset of R cannot be precise set on which a power series converges, as we shall show below.

If (b_n) is a bounded sequence of non-negative real numbers, then we define the limit superior of (b_n) to be the uniform of these numbers v such that

$$b_n \leq v \quad \text{for all sufficiently large} \quad n \in N$$

This infimum is uniquely determined and is denoted by

$$\lim sup(b_n)$$

Some other characterizations and properties of the limit superior were given in Module 2, but the only thing we need to know is

1. that if $v > \lim sup(b_n)$, then $b_n < v$ for all sufficiently large $n \in N$, and

2. that $w < \lim sup(b_n)$, then $w \leq b_n$ for infinitely many $n \in N$.

1.5 Conclusion

It is instructive to note that the convergence of an infinite series is heralded by examining the sequence of partial sums; questions concerning series of functions are answered by examining corresponding questions for sequences of functions. Therefore, our discussion on sequences of function is the plank for which the convergence of the series of function is laid. Our discussion on power series was not very elaborate because we tried to confine ourselves to elementary treatment without delving into advanced theoretical analysis. With this, we have come to the end of this course - Introduction to Real Analysis. I which you good luck.

1.6 Summary

This unit essentially covered two topics – uniform convergence of series and power series. It is important to note the similarity between sequence of functions discussed earlier and the uniform convergence of series treated in this unit.

1.7 Tutor Marked Assignments

1. If $\sum_{n=0}^{\infty} a_n x^n$ converges for $x = x_0$, prove that it converges uniformly and absolutely in the interval $|x| < |x_1|$ where $|x_1| < |x_0|$.

2. Prove that the series

$$\frac{1}{1+x^2} - \frac{1}{2+x^2} + \frac{1}{3+x^2} - \cdots$$

is uniformly convergent.

1.8 References

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